Abstract

The oscillation of neutrinos is a distinct phenomenon in particle physics; it is also the only particle physics probe we have to physics we don't understand. This new phenomenon adds 7+ parameters to the Standard Model and oscillations can probe 6 of them; measuring and understanding these 6 parameters has been the focus of an intense global effort in neutrino physics for the last two decades. In this colloquium we will explore neutrino oscillations and their behavior in the presence of background matter fields. I will present approximation schemes to better understand the relationship between the underlying parameters and the physical observables. Along the way, I will discuss several interesting math results including a historical discussion and the (re-)discovery of a linear algebra expression with broad applicability.

Neutrino Oscillations in Matter and Linear Algebra

Peter B. Denton

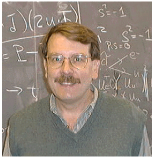
IIT Colloquium

April 22, 2021

BROOKHAVEN NATIONAL LABORATORY

Brookhaven NDI
Neutrino Discovery Initiative

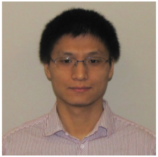
Analytic Oscillation Probability Collaborators



Stephen Parke

Hisakazu Minakata

Gabriela Barenboim



Xining Zhang

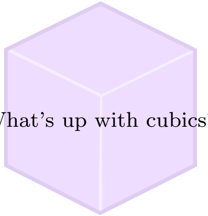
Christoph Ternes

Terrence Tao

1604.08167, 1806.01277, 1808.09453,
1902.00517, 1902.07185, 1907.02534
1908.03795, 1909.02009
github.com/PeterDenton/Nu-Pert
github.com/PeterDenton/Nu-Pert-Compare

Path

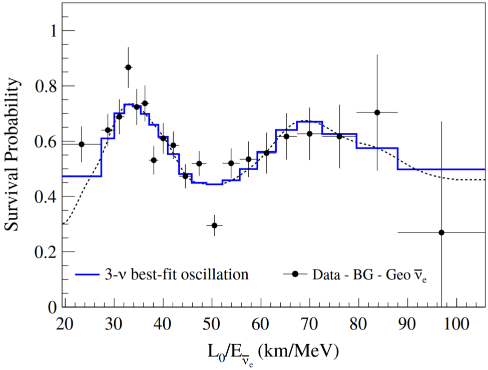
1. **Statement** of oscillation question
2. Get the eigen**values**
3. Get the eigen**vectors**
4. Useful **approx**imations



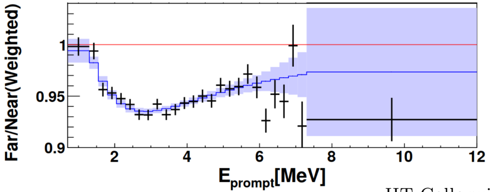
Statement of oscillation question

Neutrino Masses

1. Neutrinos experience time ⇒ must have mass
2. Neutrino oscillate ⇒ must mix & masses must be different



KamLAND 1303.4667



Daya Bay 1809.02261

Experiment to Oscillation Parameters

Six oscillation parameters: θ_{12}, θ_{13}, θ_{23}, δ, Δm^2_{21}, Δm^2_{31}

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- Atmospheric ν_μ disappearance $\to \sin 2\theta_{23}$, $|\Delta m^2_{31}|$

  SuperK, IMB, IceCube

- Solar ν_e disappearance $\to \pm\cos 2\theta_{12}$, $\pm\Delta m^2_{21}$

  SNO, Borexino, SuperK

- Reactor ν_e disappearance:
  - LBL $\to \sin 2\theta_{12}$ and $|\Delta m^2_{21}|$

    KamLAND

  - Future LBL $\to \pm\Delta m^2_{31}$

    JUNO

  - MBL $\to \theta_{13}$, $|\Delta m^2_{31}|$

    Daya Bay, RENO, Double Chooz

- Accelerator LBL ν_e appearance: $\pm\Delta m^2_{31}$, $\pm\cos 2\theta_{23}$, θ_{13}, δ

  T2K, NOvA, T2HK, DUNE

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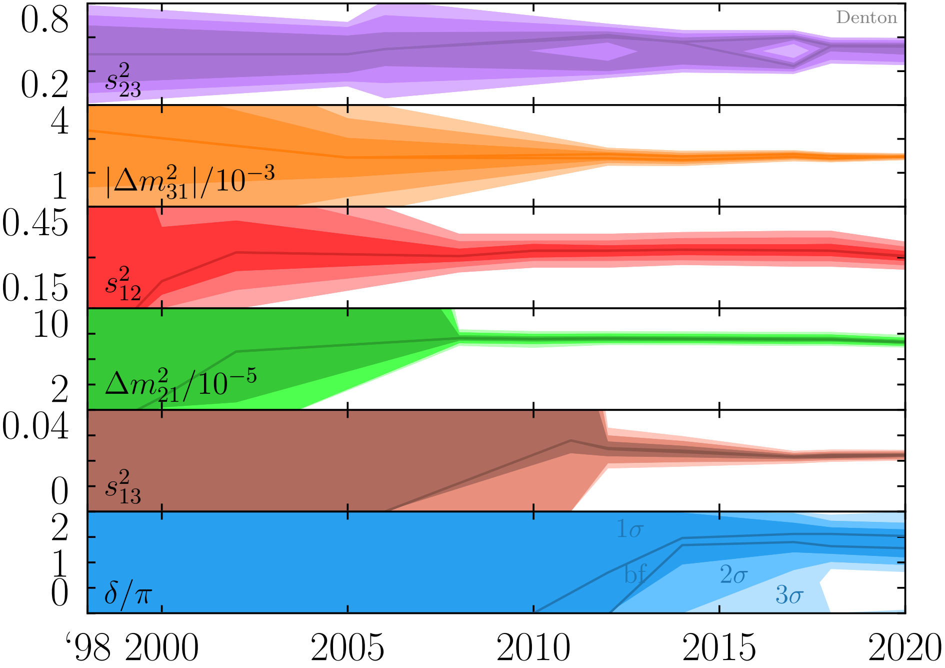
7th parameter: absolute mass scale

Cosmology, KATRIN, $0\nu\beta\beta$



Denton
0.8
0.2
s_{23}^2
4
1
$|\Delta m_{31}^2|/10^{-3}$
0.45
0.15
s_{12}^2
10
2
$\Delta m_{21}^2/10^{-5}$
0.04
0
s_{13}^2
2
1
0
δ/π
1σ
bf
2σ
3σ
‘98 2000
2005
2010
2015
2020

Schrödinger Equation

Neutrinos propagate in eigenstates of the Hamiltonian

$$i\frac{d}{dt}|\nu\rangle = H|\nu\rangle$$

In the absence of any interactions $H_{\rm vac}|\nu_i\rangle = E_i|\nu_i\rangle$.

$$|\nu_i(L)\rangle = e^{-iE_iL}|\nu_i(0)\rangle \to e^{-im_i^2/2E}|\nu_i(0)\rangle$$

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We don't produce neutrinos in eigenstates of the Hamiltonian in vacuum, e.g. mass eigenstates

$$|\nu_\alpha\rangle = \sum_{i=1}^{3} U_{\alpha i}^*|\nu_i\rangle \qquad \alpha \in \{e,\mu,\tau\}$$

U is a unitary 3×3 matrix which has four degrees of freedom

Unitarity $\Rightarrow$ 9 dofs, rephasing $\Rightarrow 9-5=4$

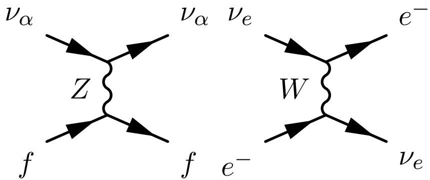
Matter Effects Matter

$$\mathcal{A}(\nu_\alpha \to \nu_\beta) = \sum_{i=1}^{3} U_{\alpha i}^* \, e^{-im_i^2 L/2E} \, U_{\beta i} \qquad\qquad P = |\mathcal{A}|^2$$

Matter Effects Matter

$$\mathcal{A}(\nu_\alpha \to \nu_\beta) = \sum_{i=1}^{3} U_{\alpha i}^* \, e^{-im_i^2 L/2E} \, U_{\beta i} \qquad P = |\mathcal{A}|^2$$

In matter ν's propagate in a new basis that depends on $a \propto N_e E_\nu$.



L. Wolfenstein PRD 17 (1978)

Eigenvalues: $m_i^2 \to \widehat{m^2}_i(a)$

Eigenvectors are given by $\theta_{ij} \to \widehat{\theta}_{ij}(a)$ $\Leftarrow$ Unitarity

Hamiltonian Dynamics

$$H_{\text{flav}} = \frac{1}{2E}\left[U\begin{pmatrix}0 & & \\ & \Delta m^2_{21} & \\ & & \Delta m^2_{31}\end{pmatrix}U^\dagger + \begin{pmatrix}a & & \\ & 0 & \\ & & 0\end{pmatrix}\right]$$

$a = 2\sqrt{2}G_F N_e E$

Hamiltonian Dynamics

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$a = 2\sqrt{2}G_F N_e E$

$$U = \begin{pmatrix}1 & & \\ & c_{23} & s_{23} \\ & -s_{23} & c_{23}\end{pmatrix}\begin{pmatrix}c_{13} & & s_{13}e^{-i\delta} \\ & 1 & \\ -s_{13}e^{i\delta} & & c_{13}\end{pmatrix}\begin{pmatrix}c_{12} & s_{12} & \\ -s_{12} & c_{12} & \\ & & 1\end{pmatrix}$$

$s_{ij} = \sin\theta_{ij}$, $c_{ij} = \cos\theta_{ij}$

PBD, R. Pestes 2006.09384

Hamiltonian Dynamics

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Find eigenvalues and eigenvectors:

PBD, R. Pestes 2006.09384

$$H_{\text{flav}} = \frac{1}{2E}\widehat{U}\begin{pmatrix}0 & & \\ & \Delta\widehat{m^2}_{21} & \\ & & \Delta\widehat{m^2}_{31}\end{pmatrix}\widehat{U}^\dagger$$

J. Kopp physics/0610206

Computationally works, but we can do better than a **black box**...

Analytic expression?

Analytic Oscillation Probabilities in Matter

- ☑ Solar: $P_{ee} \simeq \sin^2\theta_\odot$

  Approx: S. Mikheev, A. Smirnov Nuovo Cim. C9 (1986) 17-26

  Exact: S. Parke PRL 57 (1986) 2322

- ☑ Long-baseline: All three flavors

  Exact: H. Zaglauer, K. Schwarzer Z.Phys. C40 (1988) 273

  Approx: PBD, H. Minakata, S. Parke, 1604.08167

  Review: G. Barenboim, PBD, S. Parke, C. Ternes 1902.00517

- ☑ ν_e disappearance (neutrino factory):
  $\Delta\widehat{m^2}_{ee} = \widehat{m^2}_3 - (\widehat{m^2}_1 + \widehat{m^2}_2 - \Delta m^2_{21} c^2_{12})$

  PBD, S. Parke, 1808.09453

- ☐ Atmospheric

Get the eigen**values**

Eigenvalues Analytically: The Exact Solution

Solve the cubic characteristic equation: eigen**values**

$$(\widehat{m^2}_i)^3 - A(\widehat{m^2}_i)^2 + B\widehat{m^2}_i - C = 0$$

$$A \equiv \textstyle\sum_i \widehat{m^2}_i = \Delta m^2_{31} + \Delta m^2_{21} + a$$
$$B \equiv \textstyle\sum_{i>j} \widehat{m^2}_i \widehat{m^2}_j = \Delta m^2_{31}\Delta m^2_{21} + a(\Delta m^2_{ee} c^2_{13} + \Delta m^2_{21})$$
$$C \equiv \textstyle\prod_i \widehat{m^2}_i = a\Delta m^2_{31}\Delta m^2_{21} c^2_{13} c^2_{12}$$

G. Cardano *Ars Magna* 1545

V. Barger, et al. PRD 22 (1980) 2718

H. Zaglauer, K. Schwarzer Z.Phys. C40 (1988) 273

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Then write down eigen**vectors** (mixing angles)

H. Zaglauer, K. Schwarzer Z.Phys. C40 (1988) 273

K. Kimura, A. Takamura, H. Yokomakura `hep-ph/0205295`

PBD, S. Parke, X. Zhang `1907.02534`

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"Unfortunately, the algebra is rather impenetrable."

V. Barger, et al.

The Cubic

Math history aside

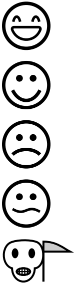
Linear: $ax + b = 0$

Quadratic: $ax^2 + bx + c = 0$

Cubic: $ax^3 + bx^2 + cx + d = 0$

Quartic: $ax^4 + bx^3 + cx^2 + dx + e = 0$

Qunitic+: $ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$



Abel-Ruffini theorem, 1824

The Cubic

Math history aside

1. **Ancients** (20-16C BC)
   Babylonians, Greeks, Chinese, Indians, Egyptians:
   thought about cubics, calculated cube roots

$$x^3 = a$$

2. Chinese **Wang Xiaotong** (7C AD):
   numerically solved 25 general cubics
3. Persian **Omar Khayyam** (11C AD):
   realized there are multiple solutions
4. Italian **Fibonacci** (12C AD):
   Approximate solution to one cubic

The Cubic

THE ITALIAN JOB

Math history aside: (16C AD)

5. Scipione del **Ferro**:
   Secret solution, nearly all (didn't know negative numbers)

$$x^3 + mx = n$$

6. Antonio **Fiore**: Ferro's student, from just before his death
7. Niccol **Tartaglia**: Claimed a solution, was challenged by Fiore
8. Gerolamo **Cardano**: Gets Tartaglia's (winner) solution, promises to keep it secret. Later publishes Ferro's solution via Fiore
9. Tartaglia challenges Cardano who denies it. Cardano's student **Ferrari** accepted, Tartaglia lost along with prestige and income
10. Cardano almost discovered complex numbers

Quartic was (nearly) solved around the same time by Ferrari, *before* the cubic solution was published

(François Viète independently solved the cubic in France a few years later)

Back to neutrinos

Eigenvalues Analytically: The Exact Solution

The cubic solution (in neutrino terms)

$$\widehat{m^2}_1 = \frac{A}{3} - \frac{1}{3}\sqrt{A^2 - 3B}S - \frac{\sqrt{3}}{3}\sqrt{A^2 - 3B}\sqrt{1 - S^2}$$

$$\widehat{m^2}_2 = \frac{A}{3} - \frac{1}{3}\sqrt{A^2 - 3B}S + \frac{\sqrt{3}}{3}\sqrt{A^2 - 3B}\sqrt{1 - S^2}$$

$$\widehat{m^2}_3 = \frac{A}{3} + \frac{2}{3}\sqrt{A^2 - 3B}S$$

$$A = \Delta m^2_{21} + \Delta m^2_{31} + a$$

$$B = \Delta m^2_{21}\Delta m^2_{31} + a\left[c^2_{13}\Delta m^2_{31} + (c^2_{12}c^2_{13} + s^2_{13})\Delta m^2_{21}\right]$$

$$C = a\Delta m^2_{21}\Delta m^2_{31}c^2_{12}c^2_{13}$$

$$S = \cos\left\{\frac{1}{3}\cos^{-1}\left[\frac{2A^3 - 9AB + 27C}{2\left(A^2 - 3B\right)^{3/2}}\right]\right\}$$

H. Zaglauer, K. Schwarzer Z.Phys. C40 (1988) 273

Get the eigen**vectors**

Values and Vectors

Probability amplitude:

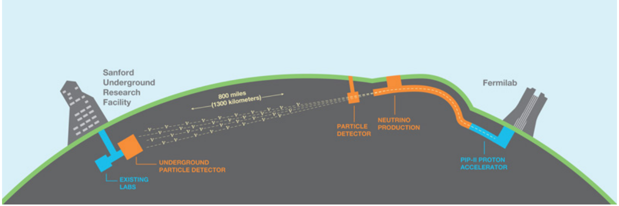
$$\mathcal{A}_{\alpha\beta} = \sum_i \widehat{U}^*_{\alpha i}\, e^{-i\widehat{m^2}_i L/2E}\, \widehat{U}_{\beta i}$$

- Eigen**values** give the frequencies of the oscillations

  Where should DUNE be?

- Eigen**vectors** give the amplitudes of the oscillations

  How many events will DUNE see?



Exact Neutrino Oscillations in Matter: Mixing Angles

$$s^2_{\widehat{12}} = \frac{-\left[(\widehat{m^2}_2)^2 - \alpha\widehat{m^2}_2 + \beta\right]\Delta\widehat{m^2}_{31}}{\left[(\widehat{m^2}_1)^2 - \alpha\widehat{m^2}_1 + \beta\right]\Delta\widehat{m^2}_{32} - \left[(\widehat{m^2}_2)^2 - \alpha\widehat{m^2}_2 + \beta\right]\Delta\widehat{m^2}_{31}}$$

$$s^2_{\widehat{13}} = \frac{(\widehat{m^2}_3)^2 - \alpha\widehat{m^2}_3 + \beta}{\Delta\widehat{m^2}_{31}\Delta\widehat{m^2}_{32}}$$

$$s^2_{\widehat{23}} = \frac{s^2_{23}E^2 + c^2_{23}F^2 + 2c_{23}s_{23}c_\delta EF}{E^2 + F^2}$$

$$e^{-i\widehat{\delta}} = \frac{c_{23}s_{23}\left(e^{-i\delta}E^2 - e^{i\delta}F^2\right) + \left(c^2_{23} - s^2_{23}\right)EF}{\sqrt{\left(s^2_{23}E^2 + c^2_{23}F^2 + 2EFc_{23}s_{23}c_\delta\right)\left(c^2_{23}E^2 + s^2_{23}F^2 - 2EFc_{23}s_{23}c_\delta\right)}}$$

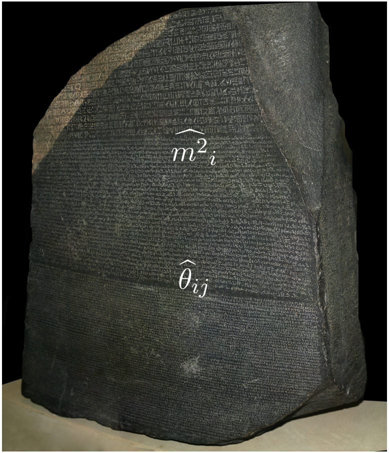
$$\alpha = c^2_{13}\Delta m^2_{31} + \left(c^2_{12}c^2_{13} + s^2_{13}\right)\Delta m^2_{21}\,, \quad \beta = c^2_{12}c^2_{13}\Delta m^2_{21}\Delta m^2_{31}$$

$$E = c_{13}s_{13}\left[\left(\widehat{m^2}_3 - \Delta m^2_{21}\right)\Delta m^2_{31} - s^2_{12}\left(\widehat{m^2}_3 - \Delta m^2_{31}\right)\Delta m^2_{21}\right]$$

$$F = c_{12}s_{12}c_{13}\left(\widehat{m^2}_3 - \Delta m^2_{31}\right)\Delta m^2_{21}$$

H. Zaglauer, K. Schwarzer Z.Phys. C40 (1988) 273

The Rosetta Stone



Eigenvalues to Eigenvectors

KTY pushed calculating the eigenvectors from the eigenvalues.

K. Kimura, A. Takamura, H. Yokomakura hep-ph/0205295

$$\widehat{U}_{\alpha i}\widehat{U}^*_{\beta i}=\frac{\widehat{p}_{\alpha\beta}\widehat{m^2}_i+\widehat{q}_{\alpha\beta}-\delta_{\alpha\beta}\widehat{m^2}_i(\widehat{m^2}_j+\widehat{m^2}_k)}{\Delta\widehat{m^2}_{ji}\Delta\widehat{m^2}_{ki}}$$

$$\widehat{p}_{\alpha\beta}=(2E)H_{\alpha\beta}$$

$$\widehat{q}_{\alpha\beta}=(2E)^2(H_{\gamma\beta}H_{\alpha\gamma}-H_{\alpha\beta}H_{\gamma\gamma})$$

valid for $\alpha\neq\beta$.

Wanted to preserve phase information for $\widehat{\delta}$.

Eigenvalues: the Rosetta Stone

We realized:

$$|\widehat{U}_{\alpha i}|^2 = \frac{(\widehat{m^2}_i - \xi_\alpha)(\widehat{m^2}_i - \chi_\alpha)}{\Delta\widehat{m^2}_{ij}\Delta\widehat{m^2}_{ik}}$$

PBD, S. Parke, X. Zhang 1907.02534

where ξ_α and χ_α are the submatrix eigenvalues

$$H = \begin{pmatrix} H_{\alpha\alpha} & H_{\alpha\beta} & H_{\alpha\gamma} \\ H_{\beta\alpha} & H_{\beta\beta} & H_{\beta\gamma} \\ H_{\gamma\alpha} & H_{\gamma\beta} & H_{\gamma\gamma} \end{pmatrix} \quad \rightarrow \quad H_\alpha = \begin{pmatrix} H_{\beta\beta} & H_{\beta\gamma} \\ H_{\gamma\beta} & H_{\gamma\gamma} \end{pmatrix}$$

e.g.

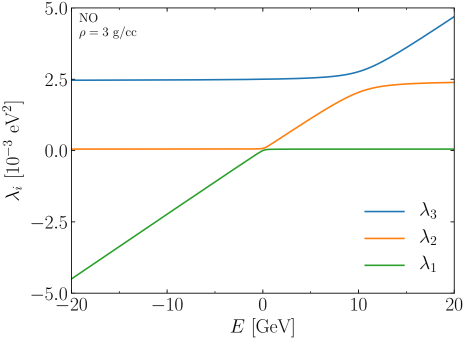
$$\xi_e + \chi_e = \Delta m^2_{21} + \Delta m^2_{ee} c^2_{13}$$

$$\xi_e \chi_e = \Delta m^2_{21}[\Delta m^2_{ee} c^2_{13} c^2_{12} + \Delta m^2_{21}(s^2_{12} c^2_{12} - s^2_{13} s^2_{12} c^2_{12})]$$

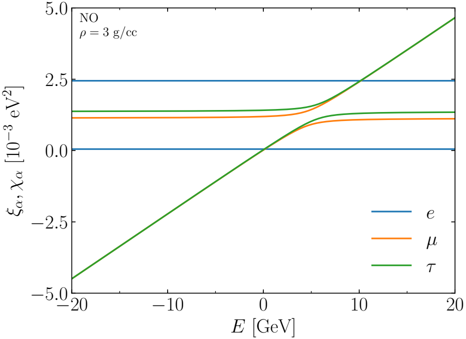
$$\Delta m^2_{ee} = c^2_{12}\Delta m^2_{31} + s^2_{12}\Delta m^2_{32}$$

H. Nunokawa, S. Parke, R. Z. Funchal hep-ph/0503283

Submatrix Eigenvalues



Eigenvalues



Submatrix Eigenvalues

Eigenvalues: the Rosetta Stone

$$s^2_{\widehat{13}} = |\widehat{U}_{e3}|^2 = \frac{(\widehat{m^2}_3 - \xi_e)(\widehat{m^2}_3 - \chi_e)}{\Delta\widehat{m^2}_{31}\Delta\widehat{m^2}_{32}}$$

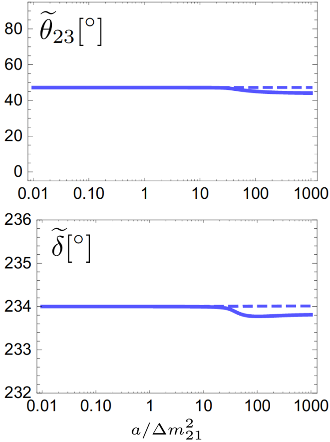
$$s^2_{\widehat{12}}c^2_{\widehat{13}} = |\widehat{U}_{e2}|^2 = -\frac{(\widehat{m^2}_2 - \xi_e)(\widehat{m^2}_2 - \chi_e)}{\Delta\widehat{m^2}_{32}\Delta\widehat{m^2}_{21}}$$

$$s^2_{\widehat{23}}c^2_{\widehat{13}} = |\widehat{U}_{\mu 3}|^2 = \frac{(\widehat{m^2}_3 - \xi_\mu)(\widehat{m^2}_3 - \chi_\mu)}{\Delta\widehat{m^2}_{31}\Delta\widehat{m^2}_{32}}$$

What about $\widehat{\delta}$?

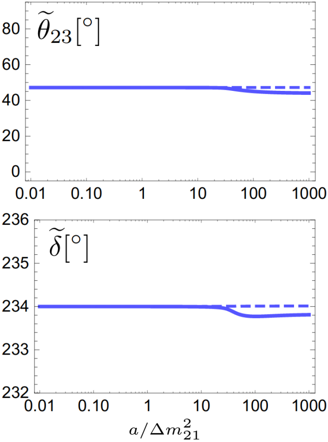
CPV From Rosetta

$\widehat{\delta}$ nearly constant, but have to get it right



Z-z. Xing, S. Zhou
Y-L. Zhou 1802.00990

CPV From Rosetta



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Toshev identity:

$$\sin\widehat{\delta} = \frac{\sin 2\theta_{23}}{\sin 2\widehat{\theta}_{23}} \sin\delta$$

S. Toshev MPL A6 (1991) 455

Get the sign of $\cos\widehat{\delta}$ from e.g. $|\widehat{U}_{\mu 1}|^2$.

In General

Two flavor:

$$|\widehat{U}_{\alpha i}|^2 = \frac{\widehat{m^2}_i - \xi_\alpha}{\Delta\widehat{m^2}_{ij}}$$

leads to

$$\sin^2\widehat{\theta} = |\widehat{U}_{e2}|^2 = \frac{\widehat{m^2}_2 - \xi_e}{\widehat{m^2}_2 - \widehat{m^2}_1}$$

$$= \frac{1}{2}\left(1 - \frac{\Delta m^2\cos 2\theta - a}{\sqrt{(\Delta m^2\cos 2\theta - a)^2 + (\Delta m^2\sin 2\theta)^2}}\right)$$

In General

Two flavor:

$$|\widehat{U}_{\alpha i}|^2 = \frac{\widehat{m^2}_i - \xi_\alpha}{\Delta\widehat{m^2}_{ij}}$$

leads to

$$\sin^2\widehat{\theta} = |\widehat{U}_{e2}|^2 = \frac{\widehat{m^2}_2 - \xi_e}{\widehat{m^2}_2 - \widehat{m^2}_1}$$

$$= \frac{1}{2}\left(1 - \frac{\Delta m^2\cos 2\theta - a}{\sqrt{(\Delta m^2\cos 2\theta - a)^2 + (\Delta m^2\sin 2\theta)^2}}\right)$$

Numerically checked for $N = 4, 5$.

True for all N?

A Cheery Firehose

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2. We emailed our result, < 2 hours later:
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"He's famously like a cheery firehose of mathematics
guess he's power-washing you today"

EIGENVECTORS FROM EIGENVALUES

PETER B. DENTON, STEPHEN J. PARKE, TERENCE TAO, AND XINING ZHANG

ABSTRACT. We present a new method of succinctly determining eigenvectors from eigenvalues. Specifically, we relate the norm squared of the elements of eigenvectors to the eigenvalues and the submatrix eigenvalues.

$$|v_{i,j}|^2 = \frac{\prod_{k=1}^{n-1}(\lambda_i - \xi_{j,k})}{\prod_{k=1;k\neq i}^{n}(\lambda_i - \lambda_k)}$$

Proofs

1. From previous result with $n-1$ subvectors using derivatives

   L. Erdos, B. Schlein, H-T. Yau 0711.1730

   T. Tao, V. Vu 0906.0510

2. Geometric formulation with exterior algebra
3. Using determinants and a Cauchy-Binet variant
4. Adjugate matrices

   Can get off-diagonal elements, thus CP phase

5. Cramer's rule
6. Two other mathematicians provided other proofs
7. Another mathematician generalized it to all square matrices

https://terrytao.wordpress.com/2019/08/13/eigenvectors-from-eigenvalues/

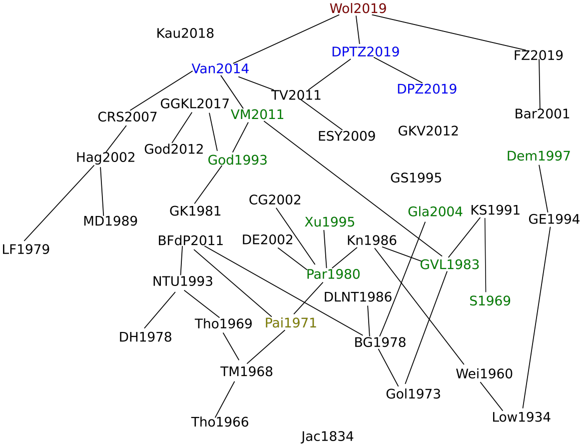
Same result previously appeared

R. Thompson, P. McEnteggert Lin. Alg. App. 1 211-243 (1968)

Historical survey

Dubbed it the: **Eigenvector-Eigenvalue Identity**

- Added many lemmas
- Reviewed existing proofs
- Discussed sociology of the disconnected citation graph



Eigenvector-Eigenvalue Identity Impact

It now appears in papers, theses, and textbooks about:

1. Neutrino oscillations
2. Radiofrequency resonators/MRI
3. Singular matrices
4. Computational methods
5. Regression analysis
6. Methods of medical research
7. Condensed matter
8. Graph theory
9. Economics of technology
10. Statistics
11. Many other math/computer science things

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11. Many other math/computer science things

Even in a field as old as linear algebra,
there are still gems hiding out there

Back to neutrinos

for good this time

Useful **approx**imations

Too "Impenetrable": Approximations

- Small matter potential: $a/\Delta m^2$

Y. Li, Y. Wang, Z-z. Xing 1605.00900

I. Martinez-Soler, H. Minakata 1904.07853

A. Khan, H. Nunokawa, S. Parke 1910.12900

- $s_{13} \sim 0.14$, $s_{13}^2 \sim 0.02$

A. Cervera, et al. hep-ph/0002108

H. Minakata 0910.5545

K. Asano, H. Minakata 1103.4387

- $\Delta m_{21}^2/\Delta m_{31}^2 \sim 0.03$

J. Arafune, J. Sato, hep-ph/9607437

A. Cervera, et al. hep-ph/0002108

M. Freund, hep-ph/0103300

E. Akhmedov, et al. hep-ph/0402175

S. Agarwalla, Y. Kao, T. Takeuchi 1302.6773

H. Minakata, S. Parke 1505.01826

PBD, H. Minakata, S. Parke 1604.08167

(See G. Barenboim, PBD, S. Parke, C. Ternes 1902.00517 for a review)

Focus On Eigenvalues

Now eigenvectors are easy enough given eigenvalues

Had previously derived approximations to both

PBD, H. Minakata, S. Parke 1604.08167

Use approximate eigenvalues in "Rosetta" formula

Rotations Are Key

Two techniques to improve precision of an approximate system:

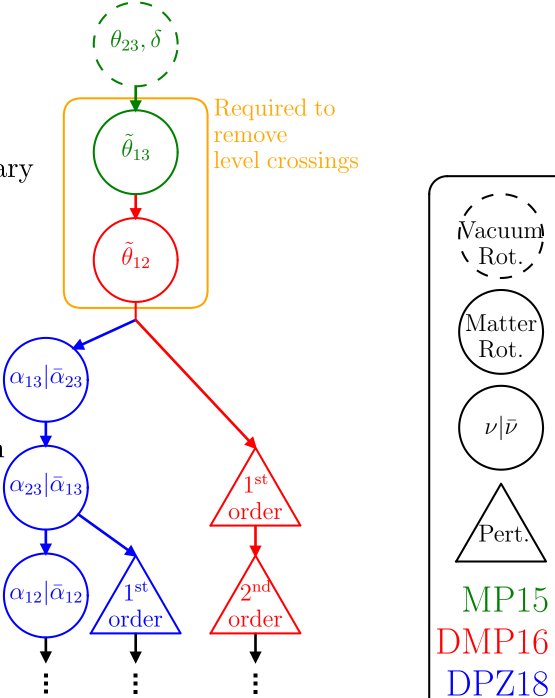
1. **Rotations**
   - Removes level crossings
   - Each step is as complicated as the last
   - Can improve the precision arbitrarily
   - Order matters: care must be taken
2. **Perturbation theory**
   - Straightforward procedure to continue ad infinitum
   - Each step is more complicated than the previous
   - Careful to avoid level crossings

We employ a hybrid approach

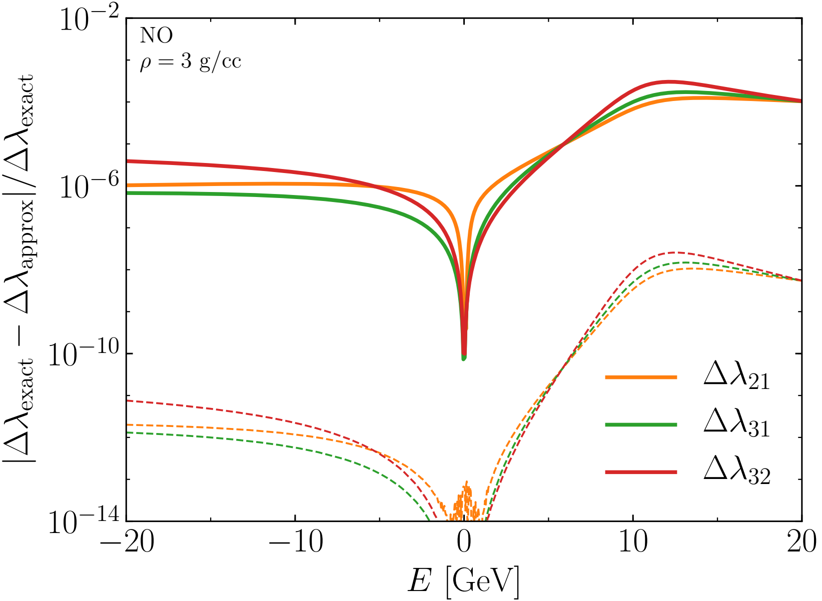
Roadmap

- Two rotations are necessary
- Order is lucky

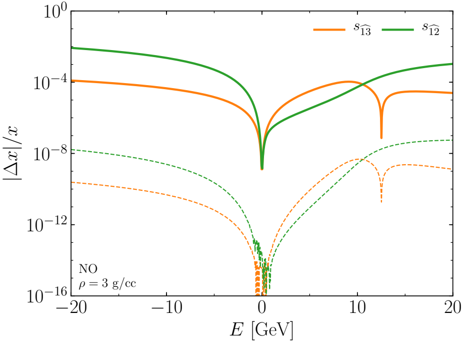
- Further precision through perturbation theory or rotations

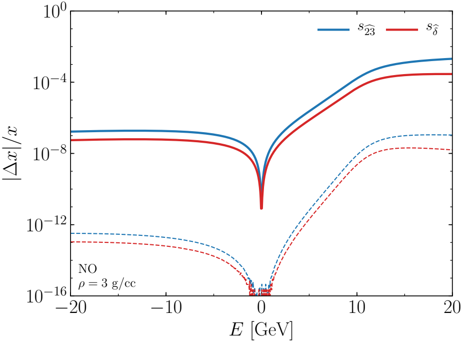


DMP Eigenvalues

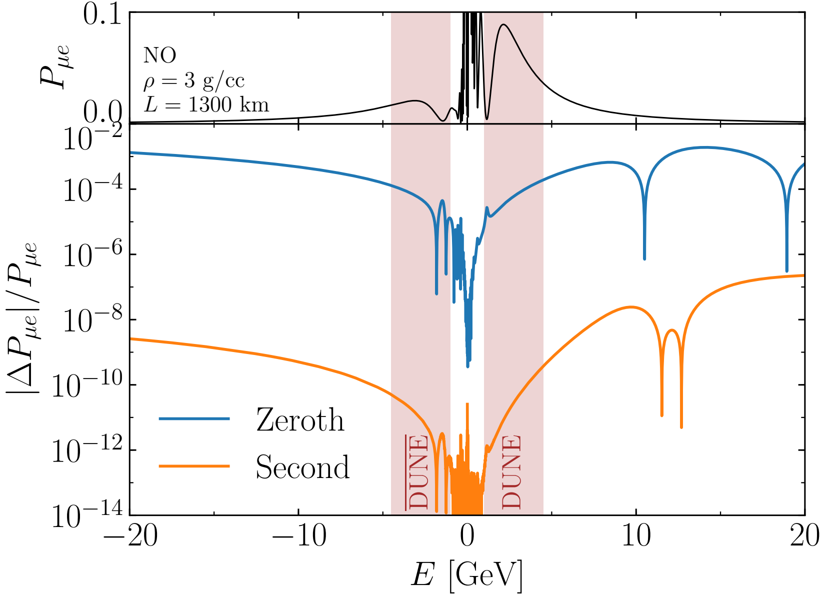


DMP Eigenvalues + Rosetta = Mixing Angles in Matter





DMP Eigenvalues + Rosetta = Probability in Matter



New Physics

DUNE and T2HK will unprecedented capabilities to test the three-neutrino oscillation picture

Extend DMP to new physics progress report:

- ☑ Sterile

  S. Parke, X. Zhang 1905.01356

- ☑ NSI

  S. Agarwalla, et al. 2103.13431

- ☐ Neutrino decay
- ☐ Decoherence
- ☐ . . .

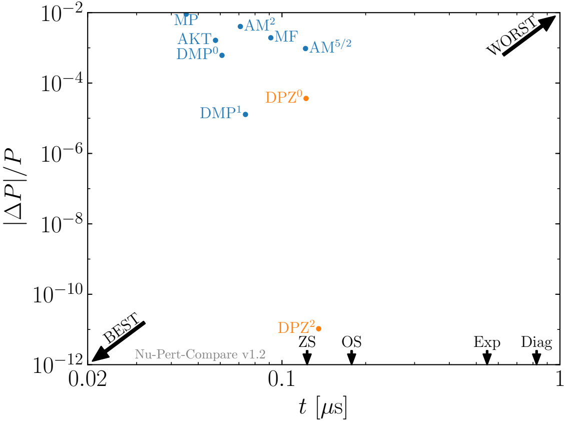
Given Rosetta, extensions should be considerably simpler

Computational Speed

NOvA's 2018 result used 54M cpu·h for Feldman Cousins, mostly on calculating oscillation probabilities

NOvA 1806.00096

news.fnal.gov/2018/07/fermilab-computing-experts-bolster-nova-evidence-1-million-cores-consumed



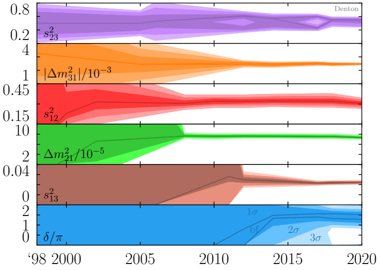
Key Points

- Long-baseline experiments are best way to probe remaining unknowns
- Matter effect requires diagonalizing Hamiltonian
- Calculate eigenvalues first
- Get eigenvectors from eigenvalues
- Approximate eigenvalues are very precise

Thanks!

Backups

References



SK hep-ex/9807003

M. Gonzalez-Garcia, et al. hep-ph/0009350

M. Maltoni, et al. hep-ph/0207227

SK hep-ex/0501064

SK hep-ex/0604011

T. Schwetz, M. Tortola, J. Valle 0808.2016

M. Gonzalez-Garcia, M. Maltoni, J. Salvado 1001.4524

T2K 1106.2822

D. Forero, M. Tortola, J. Valle 1205.4018

D. Forero, M. Tortola, J. Valle 1405.7540

P. de Salas, et al. 1708.01186

CP violation in matter

The CPV Term in Matter

The amount of CPV is

$$P_{\alpha\beta} - \bar{P}_{\alpha\beta} = \pm 16 J \sin\Delta_{21} \sin\Delta_{31} \sin\Delta_{32} \qquad \alpha \neq \beta$$

where the Jarlskog is

$$J \equiv \Im[U_{\alpha i} U_{\beta j} U^*_{\alpha j} U^*_{\beta i}] \qquad \alpha \neq \beta,\ i \neq j$$

$$J = c_{12} s_{12} c^2_{13} s_{13} c_{23} s_{23} \sin\delta$$

C. Jarlskog PRL 55 (1985)

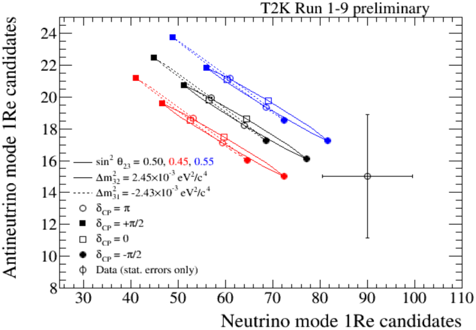
The exact term in matter is known to be

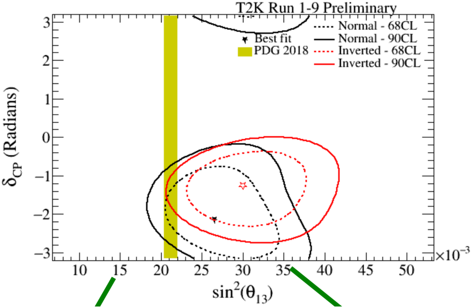
$$\frac{\widehat{J}}{J} = \frac{\Delta m^2_{21} \Delta m^2_{31} \Delta m^2_{32}}{\Delta\widehat{m^2}_{21} \Delta\widehat{m^2}_{31} \Delta\widehat{m^2}_{32}}$$

V. Naumov IJMP 1992

P. Harrison, W. Scott hep-ph/9912435

CPV Tension at T2K





$$J = c_{12}s_{12}c_{13}^2 s_{13}c_{23}s_{23}\sin\delta$$

CPV in Matter

CPV in matter can be written sans $\cos(\frac{1}{3}\cos^{-1}(\cdots))$ term.

$$\frac{\widehat{J}}{J}=\frac{\Delta m^2_{21}\Delta m^2_{31}\Delta m^2_{32}}{\Delta\widehat{m^2}_{21}\Delta\widehat{m^2}_{31}\Delta\widehat{m^2}_{32}}$$

$$\left(\Delta\widehat{m^2}_{21}\Delta\widehat{m^2}_{31}\Delta\widehat{m^2}_{32}\right)^2=(A^2-4B)(B^2-4AC)+(2AB-27C)C$$

$$A\equiv\sum_j\widehat{m^2}_j=\Delta m^2_{31}+\Delta m^2_{21}+a$$

$$B\equiv\sum_{j>k}\widehat{m^2}_j\widehat{m^2}_k=\Delta m^2_{31}\Delta m^2_{21}+a(\Delta m^2_{ee}c^2_{13}+\Delta m^2_{21})$$

$$C\equiv\prod_j\widehat{m^2}_j=a\Delta m^2_{31}\Delta m^2_{21}c^2_{13}c^2_{12}$$

This is the only oscillation quantity in matter that can be written exactly without $\cos(\frac{1}{3}\cos^{-1}(\cdots))$!

H. Yokomakura, K. Kimura, A. Takamura hep-ph/0009141

CPV Factorizes

Thus $\widehat{J}^{-2}$ is fourth order in matter potential:
only two matter corrections are needed.

$$\frac{\widehat{J}}{J} = \frac{1}{|1-(a/\alpha_1)e^{i2\theta_1}||1-(a/\alpha_2)e^{i2\theta_2}|}$$

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$$\frac{\widehat{J}}{J} = \frac{1}{|1-(a/\alpha_1)e^{i2\theta_1}||1-(a/\alpha_2)e^{i2\theta_2}|}$$

CPV in matter can be well approximated:

$$\frac{\widehat{J}}{J} \approx \frac{1}{|1-(a/\Delta m^2_{ee})e^{i2\theta_{13}}||1-(c^2_{13}a/\Delta m^2_{21})e^{i2\theta_{12}}|}$$

PBD, Parke 1902.07185

See also X. Wang, S. Zhou 1901.10882

Precise at the $< 0.04\%$ level!

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CPV Factorizes Part II

- Option 1: Use NHS identity and $\Delta\widehat{m^2}$'s
- Option 2: Use the angles?

$$\frac{\widehat{J}}{J}=\frac{s_{\widehat{12}}c_{\widehat{12}}s_{\widehat{13}}c^2_{\widehat{13}}s_{\widehat{23}}c_{\widehat{23}}\sin\widehat{\delta}}{s_{12}c_{12}s_{13}c^2_{13}s_{23}c_{23}\sin\delta}$$

Toshev: θ_{23}, δ:

S. Toshev MPL A6 (1991) 455

$$\frac{\widehat{J}}{J}=\frac{s_{\widehat{12}}c_{\widehat{12}}s_{\widehat{13}}c^2_{\widehat{13}}}{s_{12}c_{12}s_{13}c^2_{13}}$$

How to split up?

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How to split up?

From DMP:

PBD, H. Minakata, S. Parke 1604.08167

$$\frac{1}{|1-(a/\Delta m^2_{ee})e^{i2\theta_{13}}|}\approx\frac{s_{\widehat{13}}c_{\widehat{13}}}{s_{13}c_{13}}$$

Hopefully:

$$\frac{1}{|1-(c^2_{13}a/\Delta m^2_{21})e^{i2\theta_{12}}|}\approx\frac{s_{\widehat{12}}c_{\widehat{12}}c_{\widehat{13}}}{s_{12}c_{12}c_{13}}$$

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PBD, H. Minakata, S. Parke 1604.08167

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Hopefully:

$$\frac{1}{|1-(c^2_{13}a/\Delta m^2_{21})e^{i2\theta_{12}}|}\approx\frac{s_{\widehat{12}}c_{\widehat{12}}c_{\widehat{13}}}{s_{12}c_{12}c_{13}} \qquad \sim 0.4\%$$

Key Cancellation

Expect s^2_{13} or $\Delta m^2_{21}/\Delta m^2_{ee} \sim 2-3\%$ precision

The atmospheric term:

$$\Delta m^2_{31} \quad \Delta m^2_{ee} \quad \Delta m^2_{32}$$

Solar correction:

$$1 \quad c^2_{13} \quad \cos 2\theta_{13}$$

Key Cancellation

Expect s_{13}^2 or $\Delta m_{21}^2/\Delta m_{ee}^2 \sim 2-3\%$ precision

The atmospheric term:

Δm_{31}^2	Δm_{ee}^2	Δm_{32}^2
✗	✓	✗

Solar correction:

1	c_{13}^2	$\cos 2\theta_{13}$
✗	✓	✗

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The atmospheric term:

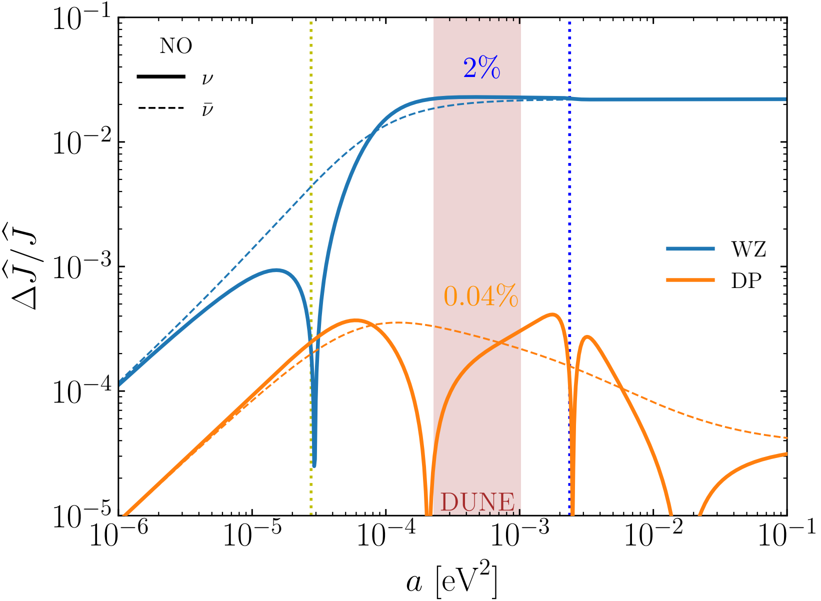
Δm_{31}^2	Δm_{ee}^2	Δm_{32}^2
✗	✓	✗

Solar correction:

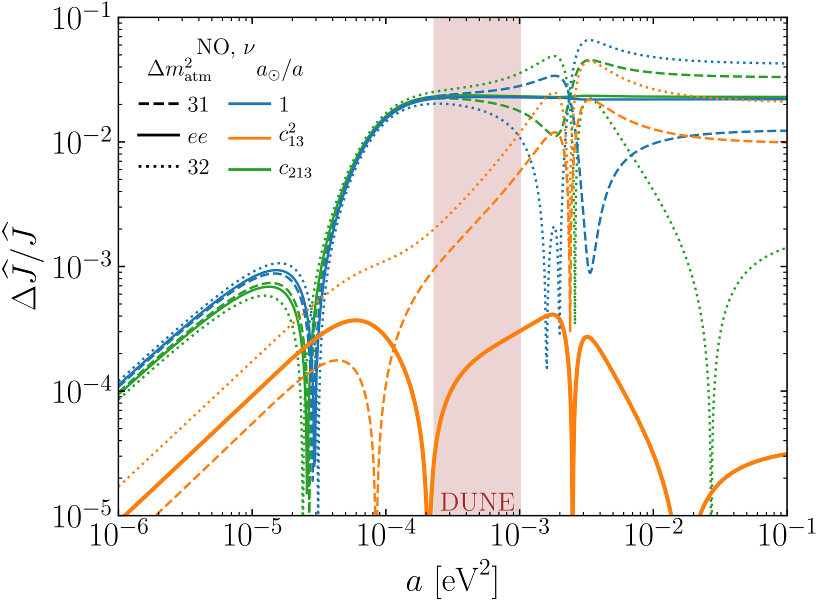
1	c_{13}^2	$\cos 2\theta_{13}$
✗	✓	✗

$$\frac{\Delta\widehat{J}}{\widehat{J}} \sim \mathcal{O}\left(s_{13}^2\frac{\Delta m_{21}^2}{\Delta m_{ee}^2}\right) + \mathcal{O}\left[\left(\frac{\Delta m_{21}^2}{\Delta m_{ee}^2}\right)^2\right] \sim 0.04\%$$

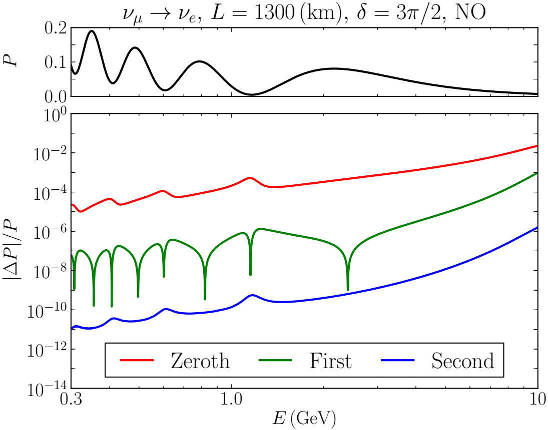
CPV In Matter Approximation Precision



Factorization Conditions



Precision



DUNE: NO, $\delta = 3\pi/2$		First min	First max
$P(\nu_\mu \to \nu_e)$		0.0047	0.081
E (GeV)		1.2	2.2
$\frac{\lvert\Delta P\rvert}{P}$	Zeroth	5×10^{-4}	4×10^{-4}
	First	3×10^{-7}	2×10^{-7}
	Second	6×10^{-10}	5×10^{-10}

Proper Expansions

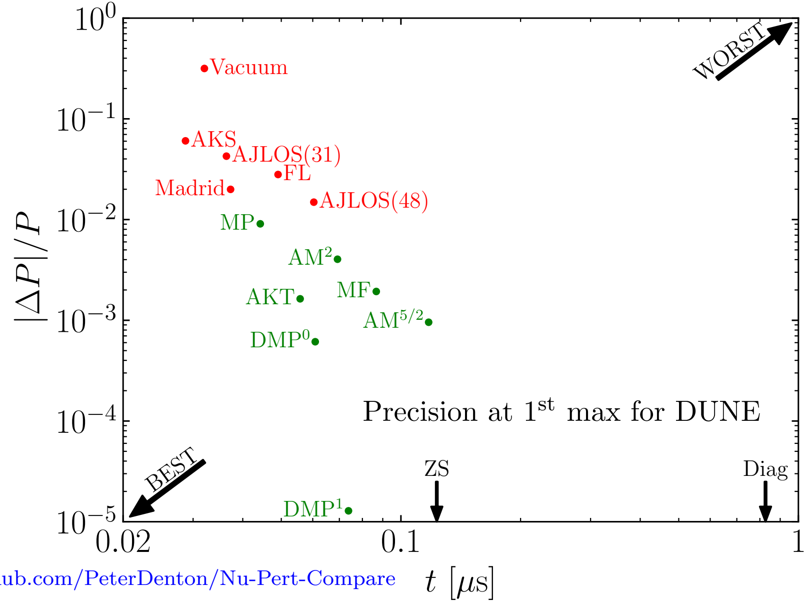
Parameter x is an <u>expansion parameter</u> iff

$$\lim_{x\to 0} P_{\text{approx}}(x) = P_{\text{exact}}(x=0)$$

	ϵ	s_{13}	$a/\Delta m^2_{31}$	
Madrid(like)	✗	✗	✗	Cervera+ hep-ph/0002108
AKT	✓	✓	✓	Agarwalla+ 1302.6773
MP	✓	✗	✗	Minakata, Parke 1505.01826
DMP	✓	✓	✓	PBD+ 1604.08167
AKS	✗	✗	✗	Arafune+ hep-ph/9703351
MF	✓	✗	✗	Freund hep-ph/0103300
AJLOS(48)	✓	✗	✗	Akhmedov+ hep-ph/0402175
AM	✗	✗	✗	Asano, Minakata 1103.4387

$$\epsilon \simeq \frac{\Delta m^2_{21}}{\Delta m^2_{ee}}$$

Speed $\propto$ Simplicity



github.com/PeterDenton/Nu-Pert-Compare

DMP Eigenvalues: Perturbative Odd Orders

After two matter rotations

$$H = \frac{1}{2E}\begin{pmatrix} \widetilde{m^2}_1 & & \\ & \widetilde{m^2}_2 & \\ & & \widetilde{m^2}_3 \end{pmatrix} + \epsilon'(a)\frac{\Delta m^2_{ee}}{2E}\begin{pmatrix} 0 & 0 & -s_{\widetilde{12}} \\ 0 & 0 & c_{\widetilde{12}} \\ -s_{\widetilde{12}} & c_{\widetilde{12}} & 0 \end{pmatrix}$$

$\epsilon' \lesssim 1\%$, $\epsilon'(a=0)=0$

Corrections to eigenvalues:

$$(\widetilde{m^2}_i)^{(1)} = (2E)H^1_{ii} = 0$$

In fact, *all* odd orders vanish

X. Zhang, PBD, S. Parke 1907.02534

Smallness parameter is actually $(\epsilon')^2 \sim 10^{-5}$!

Can be done for any 3×3 or 4×4 but not higher in general

Lots of Rotations

In a 3×3 always one zero off diagonal element:

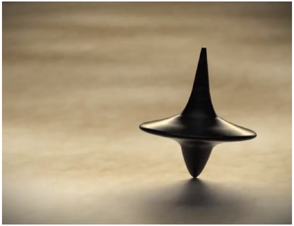
$$H_1 = \begin{pmatrix} 0 & 0 & \epsilon^a x \\ 0 & 0 & \epsilon^b y \\ \epsilon^a x^* & \epsilon^b y^* & 0 \end{pmatrix}$$

$$\epsilon \ll 1$$
$$x, y = \mathcal{O}(1)$$
$$0 < a \leq b$$

Rotate away the 1-3 term:

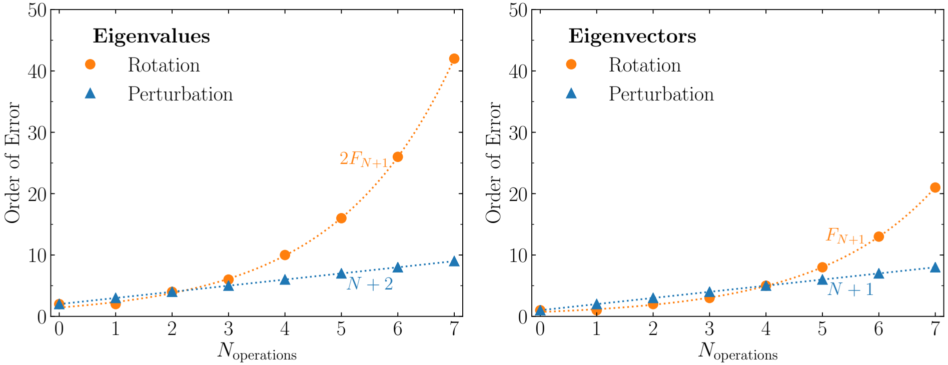
$$H_1' = \begin{pmatrix} 0 & \epsilon^{a+b} x' & 0 \\ \epsilon^{a+b} x'^* & 0 & \epsilon^b y' \\ 0 & \epsilon^b y'^* & 0 \end{pmatrix}$$

Keep on rotating large terms



Fibonacci

Continuing with rotations the error shrinks rapidly.
Take $a = b = 1$ (neutrino oscillation case)



$$F_0 \equiv 0\,, \quad F_1 \equiv 1\,, \quad F_n = F_{n-1} + F_{n-2} \text{ for } n > 1$$

$$\lim_{n\to\infty} F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$$

X. Zhang, PBD, S. Parke 1909.02009

Exponential (Fibonacci) Improvement