Abstract

Expressing neutrino oscillation probabilities in matter can be done either approximately with very good precision, or exactly with complicated expressions. Exact solutions require solving for the eigenvalues and, in turn, the eigenvectors. While there is no shortcut for the exact expressions for the eigenvalues, given those the eigenvectors can be determined in a straightforward fashion. Finally, I will show that while CPV in matter appears to extremely complicated, it is much simpler than expected both exactly and after approximations.

Recent Results in Neutrino Oscillation Theory

Peter B. Denton

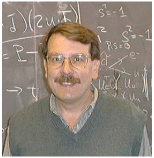
Ohio State High Energy Physics Seminar

November 4, 2019

BROOKHAVEN NATIONAL LABORATORY

Brookhaven NDI Neutrino Discovery Initiative

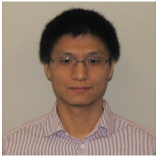
Analytic Oscillation Probability Collaborators



Stephen Parke

Hisakazu Minakata

Gabriela Barenboim



Xining Zhang

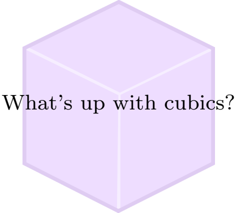
Christoph Ternes

Terrence Tao

1604.08167, 1806.01277, 1808.09453,
1902.00517, 1902.07185, 1907.02534
1908.03795, 1909.02009
github.com/PeterDenton/Nu-Pert
github.com/PeterDenton/Nu-Pert-Compare

Path

1. **Statement** of oscillation question
2. Get the eigen**values**



3. Get the eigen**vectors**
4. Useful **approx**imations
5. **CP** violation in matter

Statement of oscillation question

Experiment to Oscillation Parameters

Six oscillation parameters: θ_{12}, θ_{13}, θ_{23}, δ, Δm^2_{21}, Δm^2_{31}

- Atmospheric ν_μ disappearance $\to \sin 2\theta_{23}$, $|\Delta m^2_{31}|$

  SuperK, IMB, IceCube
- Solar ν_e disappearance $\to \pm\cos 2\theta_{12}$, $\pm\Delta m^2_{21}$

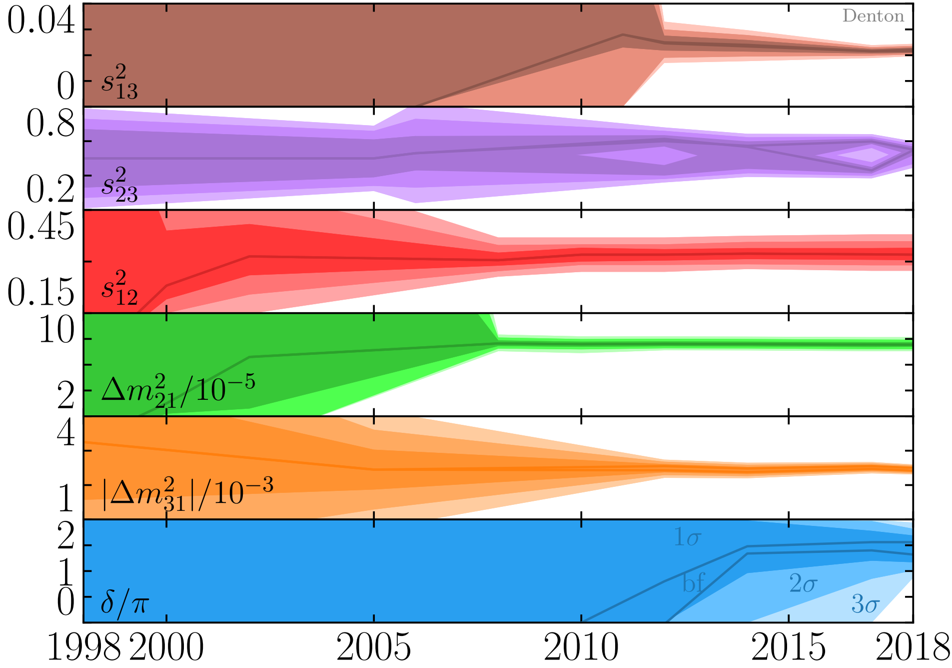
  SNO, Borexino, SuperK
- Reactor ν_e disappearance:
  - LBL $\to \sin 2\theta_{12}$ and $|\Delta m^2_{21}|$

    KamLand
  - Future LBL $\to \pm\Delta m^2_{31}$

    JUNO
  - MBL $\to \theta_{13}$, $|\Delta m^2_{31}|$

    Daya Bay, RENO, Double Chooz
- Accelerator LBL ν_e appearance: $\pm\Delta m^2_{31}$, $\pm\cos 2\theta_{23}$, θ_{13}, δ

  T2K, NOvA, T2HK, DUNE


Denton
0.04
0
s_{13}^2
0.8
0.2
s_{23}^2
0.45
0.15
s_{12}^2
10
2
$\Delta m_{21}^2/10^{-5}$
4
1
$|\Delta m_{31}^2|/10^{-3}$
2
1
0
δ/π
1σ
bf
2σ
3σ
1998
2000
2005
2010
2015
2018

What is $P(\nu_\mu \to \nu_e)$?

$$P(\overset{(-)}{\nu}_\mu \to \overset{(-)}{\nu}_e) = |\mathcal{A}_{\mu e}|^2 \qquad \mathcal{A}_{\mu e} = \mathcal{A}_{31} + e^{\pm i\Delta_{32}}\mathcal{A}_{21}$$

$$\mathcal{A}_{31} = 2s_{13}c_{13}s_{23}\sin\Delta_{31}$$

$$\mathcal{A}_{21} = 2s_{12}c_{13}(c_{12}c_{23}e^{i\delta} - s_{12}s_{13}s_{23})\sin\Delta_{21}$$

$\Delta_{ij} = \Delta m^2{}_{ij}L/4E$

What is $P(\nu_\mu \to \nu_e)$?

$$P(\overset{(-)}{\nu}_\mu \to \overset{(-)}{\nu}_e) = |\mathcal{A}_{\mu e}|^2 \qquad \mathcal{A}_{\mu e} = \mathcal{A}_{31} + e^{\pm i\Delta_{32}}\mathcal{A}_{21}$$

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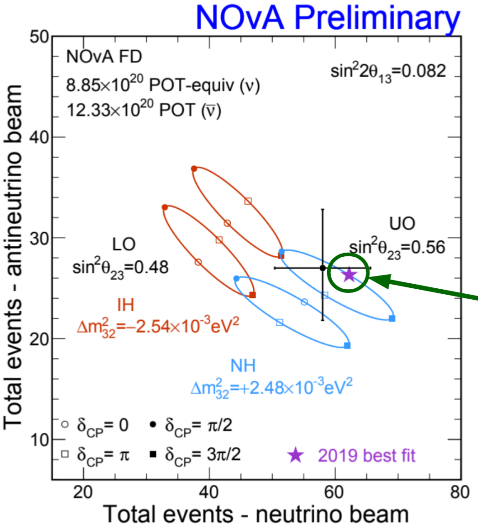
$$\Delta_{ij} = \Delta m^2{}_{ij}L/4E$$

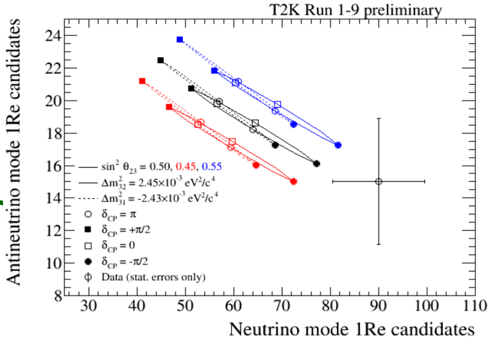
. . . in matter?

Now: NOvA, T2K, MINOS, . . .
Upcoming: DUNE, T2HK, . . .
Second maximum: T2HKK? ESSnuSB? . . .
nuSTORM?

Biprobability



810 km



295 km

Analytic Oscillation Probabilities in Matter

- ☑ Solar: $P_{ee} \simeq \sin^2\theta_\odot$

  Approx: S. Mikheev, A. Smirnov Nuovo Cim. C9 (1986) 17-26

  Exact: S. Parke PRL 57 (1986) 2322

- ☑ Long-baseline: All three flavors

  Exact: H. Zaglauer, K. Schwarzer Z.Phys. C40 (1988) 273

  Approx: PBD, H. Minakata, S. Parke, 1604.08167

- ☑ ν_e disappearance (neutrino factory):
  $\Delta\widehat{m^2}_{ee} = \widehat{m^2}_3 - (\widehat{m^2}_1 + \widehat{m^2}_2 - \Delta m^2_{21} c^2_{12})$

  PBD, S. Parke, 1808.09453

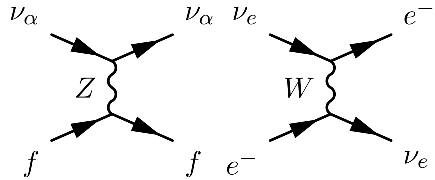
- ☐ Atmospheric

Matter Effects Matter

Call Schrödinger equation's eigenvalues m_i^2 and eigenvectors U_i.

$$\mathcal{A}(\nu_\alpha \to \nu_\beta) = \sum_{i=1}^{3} U_{\alpha i}^* U_{\beta i} e^{-im_i^2 L/2E} \qquad P = |\mathcal{A}|^2$$

In matter ν's propagate in a new basis that depends on $a \propto \rho E$.



L. Wolfenstein PRD 17 (1978)

Eigenvalues: $m_i^2 \to \widehat{m^2}_i(a)$
Eigenvectors are given by $\theta_{ij} \to \widehat{\theta}_{ij}(a) \qquad \Leftarrow \qquad$ Unitarity

Hamiltonian Dynamics

$$H = \frac{1}{2E}\left[U\begin{pmatrix}0 & & \\ & \Delta m^2_{21} & \\ & & \Delta m^2_{31}\end{pmatrix}U^\dagger + \begin{pmatrix}a & & \\ & 0 & \\ & & 0\end{pmatrix}\right]$$

$a = 2\sqrt{2}G_F N_e E$

$$U = \begin{pmatrix}1 & & \\ & c_{23} & s_{23} \\ & -s_{23} & c_{23}\end{pmatrix}\begin{pmatrix}c_{13} & & s_{13}e^{-i\delta} \\ & 1 & \\ -s_{13}e^{i\delta} & & c_{13}\end{pmatrix}\begin{pmatrix}c_{12} & s_{12} & \\ -s_{12} & c_{12} & \\ & & 1\end{pmatrix}$$

Find eigenvalues and eigenvectors:

$$H = \frac{1}{2E}\widehat{U}\begin{pmatrix}0 & & \\ & \Delta\widehat{m^2}_{21} & \\ & & \Delta\widehat{m^2}_{31}\end{pmatrix}\widehat{U}^\dagger$$

J. Kopp physics/0610206

Computationally works, but we can do better than a **black box**...
Analytic expression?

Get the eigen**values**

Eigenvalues Analytically: The Exact Solution

Solve the cubic characteristic equation: eigen**values**

G. Cardano *Ars Magna* 1545

V. Barger, et al. PRD 22 (1980) 2718

H. Zaglauer, K. Schwarzer Z.Phys. C40 (1988) 273

Then write down eigen**vectors** (mixing angles)

H. Zaglauer, K. Schwarzer Z.Phys. C40 (1988) 273

K. Kimura, A. Takamura, H. Yokomakura hep-ph/0205295

PBD, S. Parke, X. Zhang 1907.02534

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PBD, S. Parke, X. Zhang `1907.02534`

"Unfortunately, the algebra is rather impenetrable."

V. Barger, et al.

The Cubic

Math history aside

Linear:	$ax + b = 0$
Quadratic:	$ax^2 + bx + c = 0$
Cubic:	$ax^3 + bx^2 + cx + d = 0$
Quartic:	$ax^4 + bx^3 + cx^2 + dx + e = 0$
Qunitic+:	$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$

Abel-Ruffini theorem, 1824

Eigenvalues Analytically: The Exact Solution

The cubic solution (in neutrino terms)

$$\widehat{m^2}_1 = \frac{A}{3} - \frac{1}{3}\sqrt{A^2-3B}S - \frac{\sqrt{3}}{3}\sqrt{A^2-3B}\sqrt{1-S^2}$$

$$\widehat{m^2}_2 = \frac{A}{3} - \frac{1}{3}\sqrt{A^2-3B}S + \frac{\sqrt{3}}{3}\sqrt{A^2-3B}\sqrt{1-S^2}$$

$$\widehat{m^2}_3 = \frac{A}{3} + \frac{2}{3}\sqrt{A^2-3B}S$$

$$A = \Delta m^2_{21} + \Delta m^2_{31} + a$$

$$B = \Delta m^2_{21}\Delta m^2_{31} + a\left[c^2_{13}\Delta m^2_{31} + (c^2_{12}c^2_{13} + s^2_{13})\Delta m^2_{21}\right]$$

$$C = a\Delta m^2_{21}\Delta m^2_{31}c^2_{12}c^2_{13}$$

$$S = \cos\left\{\frac{1}{3}\cos^{-1}\left[\frac{2A^3 - 9AB + 27C}{2\left(A^2-3B\right)^{3/2}}\right]\right\}$$

Get the eigen**vectors**

Values and Vectors

Probability amplitude:

$$\mathcal{A}_{\alpha\beta} = \sum_i \widehat{U}^*_{\alpha i} \widehat{U}_{\beta i} \; e^{-i\widehat{m^2}_i L/2E}$$

- Eigen**values** give the frequencies of the oscillations

  Where should DUNE be?

- Eigen**vectors** give the amplitudes of the oscillations

  How many events will DUNE see?

Exact Neutrino Oscillations in Matter: Mixing Angles

$$s^2_{\widehat{12}} = \frac{-\left[(\widehat{m^2}_2)^2 - \alpha\widehat{m^2}_2 + \beta\right]\Delta\widehat{m^2}_{31}}{\left[(\widehat{m^2}_1)^2 - \alpha\widehat{m^2}_1 + \beta\right]\Delta\widehat{m^2}_{32} - \left[(\widehat{m^2}_2)^2 - \alpha\widehat{m^2}_2 + \beta\right]\Delta\widehat{m^2}_{31}}$$

$$s^2_{\widehat{13}} = \frac{(\widehat{m^2}_3)^2 - \alpha\widehat{m^2}_3 + \beta}{\Delta\widehat{m^2}_{31}\Delta\widehat{m^2}_{32}}$$

$$s^2_{\widehat{23}} = \frac{s^2_{23}E^2 + c^2_{23}F^2 + 2c_{23}s_{23}c_\delta EF}{E^2 + F^2}$$

$$e^{-i\widehat{\delta}} = \frac{c_{23}s_{23}\left(e^{-i\delta}E^2 - e^{i\delta}F^2\right) + \left(c^2_{23} - s^2_{23}\right)EF}{\sqrt{\left(s^2_{23}E^2 + c^2_{23}F^2 + 2EFc_{23}s_{23}c_\delta\right)\left(c^2_{23}E^2 + s^2_{23}F^2 - 2EFc_{23}s_{23}c_\delta\right)}}$$

$$\alpha = c^2_{13}\Delta m^2_{31} + \left(c^2_{12}c^2_{13} + s^2_{13}\right)\Delta m^2_{21}\,, \quad \beta = c^2_{12}c^2_{13}\Delta m^2_{21}\Delta m^2_{31}$$

$$E = c_{13}s_{13}\left[\left(\widehat{m^2}_3 - \Delta m^2_{21}\right)\Delta m^2_{31} - s^2_{12}\left(\widehat{m^2}_3 - \Delta m^2_{31}\right)\Delta m^2_{21}\right]$$

$$F = c_{12}s_{12}c_{13}\left(\widehat{m^2}_3 - \Delta m^2_{31}\right)\Delta m^2_{21}$$

H. Zaglauer, K. Schwarzer Z.Phys. C40 (1988) 273

Too "Impenetrable": Approximations

- Small matter potential: $a/\Delta m^2$

Y. Li, Y. Wang, Z-z. Xing 1605.00900

I. Martinez-Soler, H. Minakata 1904.07853

A. Khan, H. Nunokawa, S. Parke 1910.12900

- $s_{13} \sim 0.14$, $s_{13}^2 \sim 0.02$

A. Cervera, et al. hep-ph/0002108

H. Minakata 0910.5545

K. Asano, H. Minakata 1103.4387

- $\Delta m_{21}^2/\Delta m_{31}^2 \sim 0.03$

J. Arafune, J. Sato, hep-ph/9607437

A. Cervera, et al. hep-ph/0002108

M. Freund, hep-ph/0103300

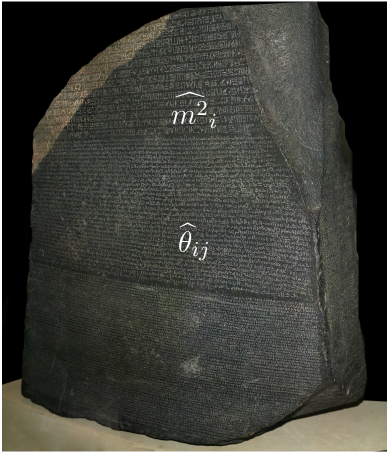
E. Akhmedov, et al. hep-ph/0402175

S. Agarwalla, Y. Kao, T. Takeuchi 1302.6773

H. Minakata, S. Parke 1505.01826

PBD, H. Minakata, S. Parke 1604.08167

(See G. Barenboim, PBD, S. Parke, C. Ternes 1902.00517 for a review)



$\widehat{m^2}_i$
$\widehat{\theta}_{ij}$

Eigenvalues to Eigenvectors

KTY pushed calculating the eigenvectors from the eigenvalues.

K. Kimura, A. Takamura, H. Yokomakura hep-ph/0205295

$$\widehat{U}_{\alpha i}\widehat{U}^*_{\beta i} = \frac{\widehat{p}_{\alpha\beta}\widehat{m^2}_i + \widehat{q}_{\alpha\beta} - \delta_{\alpha\beta}\widehat{m^2}_i(\widehat{m^2}_j + \widehat{m^2}_k)}{\Delta\widehat{m^2}_{ji}\Delta\widehat{m^2}_{ki}}$$

$\widehat{p}_{\alpha\beta} = (2E)H_{\alpha\beta}$

$\widehat{q}_{\alpha\beta} = \sum_{i<j}\widehat{m^2}_i\widehat{m^2}_j\widehat{U}_{\alpha k}\widehat{U}^*_{\beta k}$ for $k \neq i, j$

valid for $\alpha \neq \beta$.

Wanted to preserve phase information for $\widehat{\delta}$.

Eigenvalues: the Rosetta Stone

We realized:

$$|\widehat{U}_{\alpha i}|^2 = \frac{(\widehat{m^2}_i - \xi_\alpha)(\widehat{m^2}_i - \chi_\alpha)}{\Delta\widehat{m^2}_{ij}\Delta\widehat{m^2}_{ik}}$$

PBD, S. Parke, X. Zhang 1907.02534

where ξ_α and χ_α are the submatrix eigenvalues of H_α

$$H = \begin{pmatrix} H_{\alpha\alpha} & H_{\alpha\beta} & H_{\alpha\gamma} \\ H_{\beta\alpha} & H_{\beta\beta} & H_{\beta\gamma} \\ H_{\gamma\alpha} & H_{\gamma\beta} & H_{\gamma\gamma} \end{pmatrix} \quad \to \quad H_\alpha = \begin{pmatrix} H_{\beta\beta} & H_{\beta\gamma} \\ H_{\gamma\beta} & H_{\gamma\gamma} \end{pmatrix}$$

e.g.

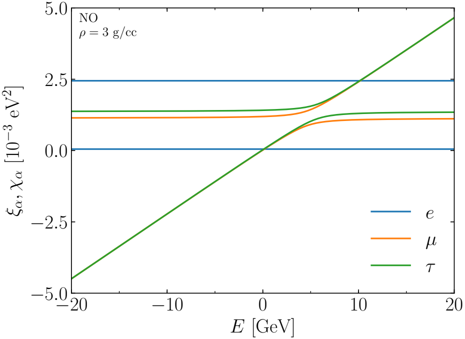
$$\xi_e + \chi_e = \Delta m^2_{21} + \Delta m^2_{ee} c^2_{13}$$

$$\xi_e \chi_e = \Delta m^2_{21}[\Delta m^2_{ee} c^2_{13} c^2_{12} + \Delta m^2_{21}(s^2_{12} c^2_{12} - s^2_{13} s^2_{12} c^2_{12})]$$

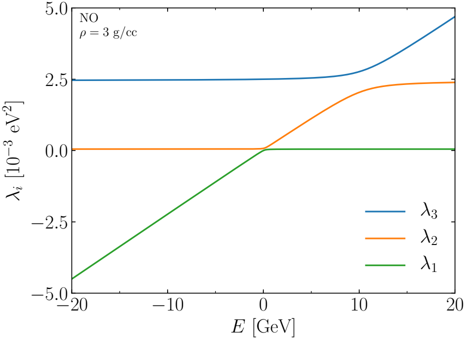
$$\Delta m^2_{ee} = c^2_{12}\Delta m^2_{31} + s^2_{12}\Delta m^2_{32}$$

H. Nunokawa, S. Parke, R. Z. Funchal hep-ph/0503283

Submatrix Eigenvalues



Submatrix Eigenvalues



Eigenvalues

Eigenvalues: the Rosetta Stone

$$s^2_{\widehat{13}} = |\widehat{U}_{e3}|^2 = \frac{(\widehat{m^2}_3 - \xi_e)(\widehat{m^2}_3 - \chi_e)}{\Delta\widehat{m^2}_{31}\Delta\widehat{m^2}_{32}}$$

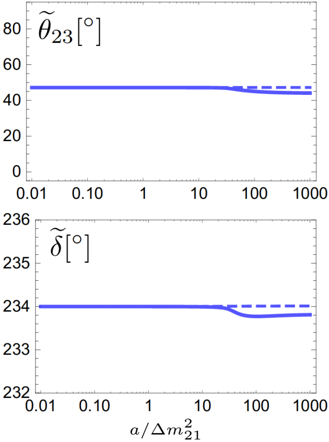
$$s^2_{\widehat{12}}c^2_{\widehat{13}} = |\widehat{U}_{e2}|^2 = -\frac{(\widehat{m^2}_2 - \xi_e)(\widehat{m^2}_2 - \chi_e)}{\Delta\widehat{m^2}_{32}\Delta\widehat{m^2}_{21}}$$

$$s^2_{\widehat{23}}c^2_{\widehat{13}} = |\widehat{U}_{\mu 3}|^2 = \frac{(\widehat{m^2}_3 - \xi_\mu)(\widehat{m^2}_3 - \chi_\mu)}{\Delta\widehat{m^2}_{31}\Delta\widehat{m^2}_{32}}$$

What about $\widehat{\delta}$?

CPV From Rosetta

$\widehat{\delta}$ nearly constant, but have to get it right



Z-z. Xing, S. Zhou
Y-L. Zhou 1802.00990

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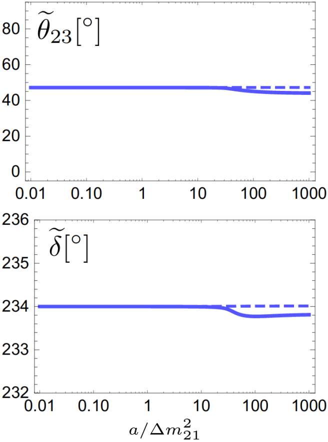

Z-z. Xing, S. Zhou
Y-L. Zhou 1802.00990

Toshev identity:

$$\sin\widehat{\delta} = \frac{\sin 2\theta_{23}}{\sin 2\widehat{\theta}_{23}}\sin\delta$$

S. Toshev MPL A6 (1991) 455

Get the sign of $\cos\widehat{\delta}$ from e.g. $|\widehat{U}_{\mu 1}|^2$.

In General

Two flavor:

$$|\widehat{U}_{\alpha i}|^2 = \frac{\widehat{m^2}_i - \xi_\alpha}{\Delta\widehat{m^2}_{ij}}$$

leads to

$$\sin^2\widehat{\theta} = |\widehat{U}_{e2}|^2 = \frac{\widehat{m^2}_2 - \xi_e}{\widehat{m^2}_2 - \widehat{m^2}_1}$$

$$= \frac{1}{2}\left(1 - \frac{\Delta m^2\cos 2\theta - a}{\sqrt{(\Delta m^2\cos 2\theta - a)^2 + (\Delta m^2\sin 2\theta)^2}}\right)$$

In General

Two flavor:

$$|\widehat{U}_{\alpha i}|^2 = \frac{\widehat{m^2}_i - \xi_\alpha}{\Delta\widehat{m^2}_{ij}}$$

leads to

$$\sin^2\widehat{\theta} = |\widehat{U}_{e2}|^2 = \frac{\widehat{m^2}_2 - \xi_e}{\widehat{m^2}_2 - \widehat{m^2}_1}$$

$$= \frac{1}{2}\left(1 - \frac{\Delta m^2\cos 2\theta - a}{\sqrt{(\Delta m^2\cos 2\theta - a)^2 + (\Delta m^2\sin 2\theta)^2}}\right)$$

Numerically checked for $N = 4, 5$.

True for all N?

A Cheery Firehose

1. Terry posted on the same question with a different answer
2. We emailed our result, < 2 hours later:
   - "Very nice identity!"
   - New result
   - 3 distinct proofs
3. 6d later, we've sorted 1 proof, send a draft, < 1 hr later:
   - Agrees to a paper
   - Adds a corollary
   - Adds several new observation
4. Barely processed that, another email $< \frac{1}{2}$ day later
   - A more general proof
5. We sent confirmation that the v_i are normed
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"He's famously like a cheery firehose of mathematics
Guess he's power-washing you today"

EIGENVECTORS FROM EIGENVALUES

PETER B. DENTON, STEPHEN J. PARKE, TERENCE TAO, AND XINING ZHANG

ABSTRACT. We present a new method of succinctly determining eigenvectors from eigenvalues. Specifically, we relate the norm squared of the elements of eigenvectors to the eigenvalues and the submatrix eigenvalues.

$$|v_{i,j}|^2 = \frac{\prod_{k=1}^{n-1}(\lambda_i - \xi_{j,k})}{\prod_{k=1;k\neq i}^{n}(\lambda_i - \lambda_k)}$$

Proofs

1. From previous result with $n-1$ subvectors using derivatives

   L. Erdos, B. Schlein, H-T. Yau 0711.1730

   T. Tao, V. Vu 0906.0510
2. Geometric formulation with exterior algebra
3. Using determinants and a Cauchy-Binet variant
4. Adjugate matrices

   Can get off-diagonal elements, thus CP phase
5. Cramer's rule
6. Two other mathematicians provided other proofs
7. Another mathematician generalized it to all square matrices

https://terrytao.wordpress.com/2019/08/13/eigenvectors-from-eigenvalues/

Useful **approx**imations

Focus On Eigenvalues

Now eigenvectors are easy enough given eigenvalues

Had previously derived approximations to both

PBD, H. Minakata, S. Parke 1604.08167

Use approximate eigenvalues in "Rosetta" formula

Rotations Are Key

Two techniques to improve precision of an approximate system:

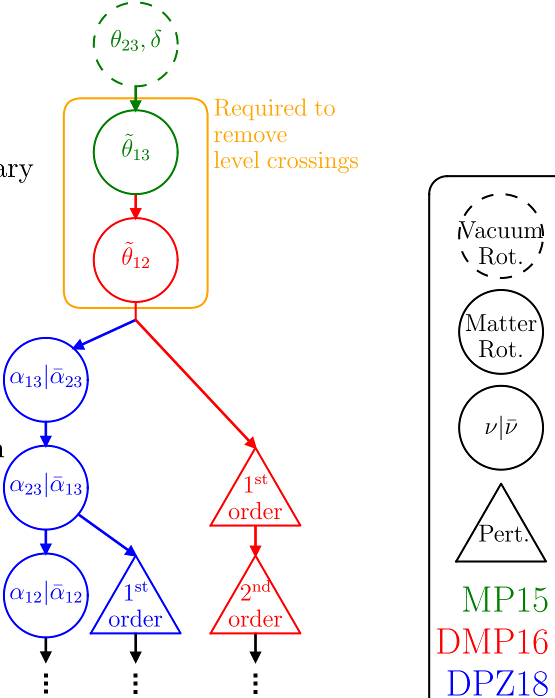
1. **Perturbation theory**
   - Straightforward procedure to continue ad infinitum
   - Each step is more complicated than the previous
   - Careful to avoid level crossings
2. **Rotations**
   - Removes level crossings
   - Each step is as complicated as the last
   - Can improve the precision arbitrarily
   - Order matters: care must be taken

We employ a hybrid approach

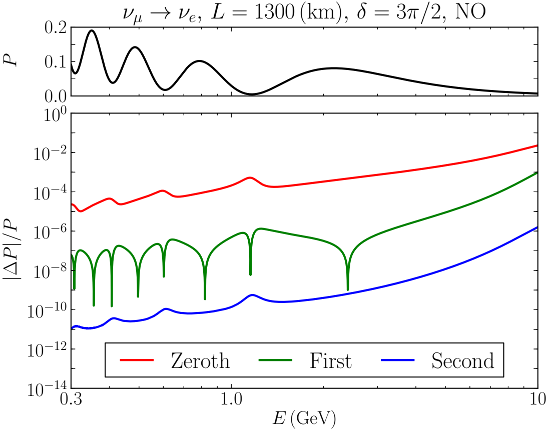
Roadmap

- Two rotations are necessary
- Order is lucky

- Further precision through perturbation theory or rotations

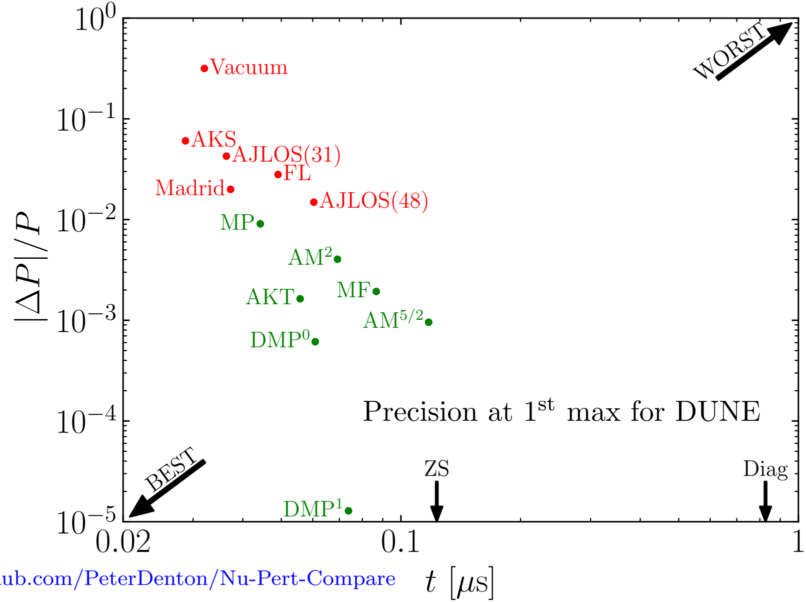


Precision



DUNE: NO, $\delta = 3\pi/2$		First min	First max
$P(\nu_\mu \to \nu_e)$		0.0047	0.081
E (GeV)		1.2	2.2
$\frac{\|\Delta P\|}{P}$	Zeroth	5×10^{-4}	4×10^{-4}
	First	3×10^{-7}	2×10^{-7}
	Second	6×10^{-10}	5×10^{-10}

Speed ∝ Simplicity



github.com/PeterDenton/Nu-Pert-Compare

DMP Eigenvalues: Perturbative Odd Orders

After two matter rotations

$$H = \frac{1}{2E}\begin{pmatrix} \widetilde{m^2}_1 & & \\ & \widetilde{m^2}_2 & \\ & & \widetilde{m^2}_3 \end{pmatrix} + \epsilon'(a)\frac{\Delta m^2_{ee}}{2E}\begin{pmatrix} 0 & 0 & -s_{\widetilde{12}} \\ 0 & 0 & c_{\widetilde{12}} \\ -s_{\widetilde{12}} & c_{\widetilde{12}} & 0 \end{pmatrix}$$

$\epsilon' \lesssim 1\%$, $\epsilon'(a=0)=0$

Corrections to eigenvalues:

$$(\widetilde{m^2}_i)^{(1)} = (2E)H^1_{ii} = 0$$

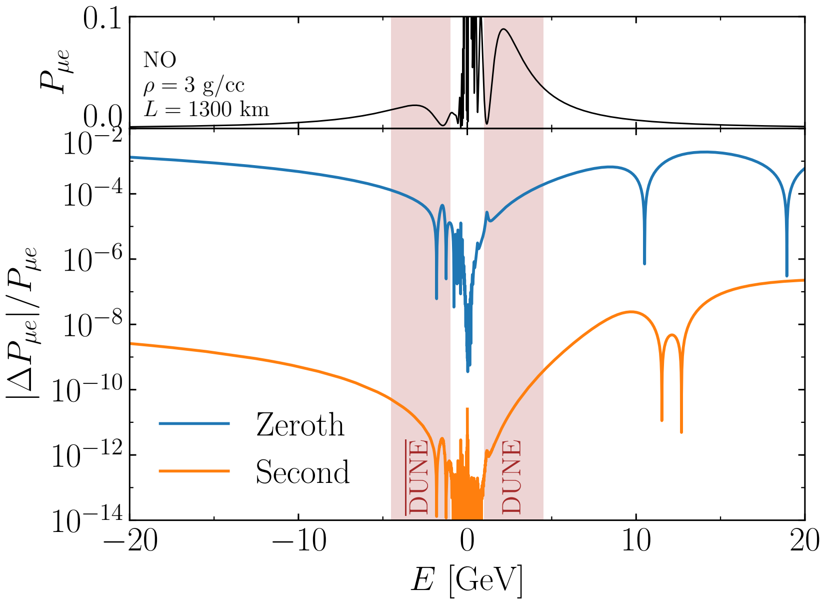
In fact, *all* odd orders vanish

X. Zhang, PBD, S. Parke 1907.02534

Smallness parameter is actually $(\epsilon')^2 \sim 10^{-5}$!

Can be done for any 3×3 or 4×4 but not higher in general

DMP Eigenvalues + Rosetta



Lots of Rotations

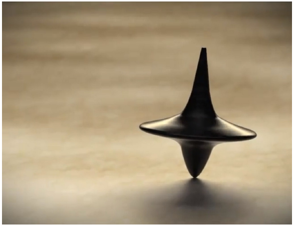
In a 3×3 always one zero off diagonal element:

$$H_1 = \begin{pmatrix} 0 & 0 & \epsilon^a x \\ 0 & 0 & \epsilon^b y \\ \epsilon^a x^* & \epsilon^b y^* & 0 \end{pmatrix} \qquad \begin{matrix} \epsilon \ll 1 \\ x, y = \mathcal{O}(1) \\ 0 < a \leq b \end{matrix}$$

Rotate away the 1-3 term:

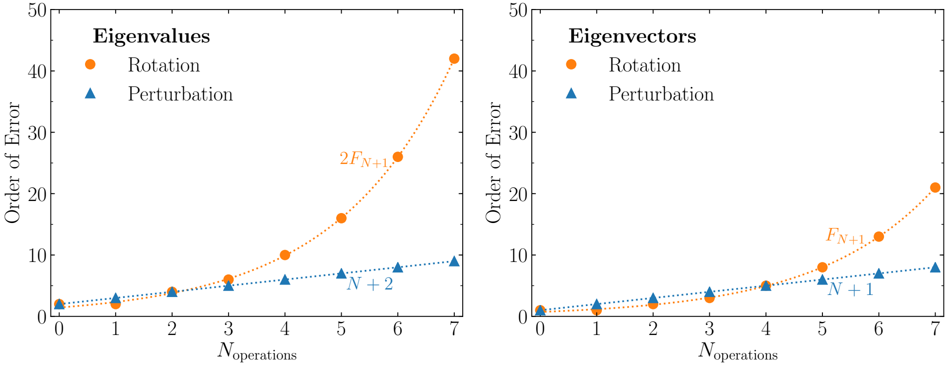
$$H_1' = \begin{pmatrix} 0 & \epsilon^{a+b} x' & 0 \\ \epsilon^{a+b} x'^* & 0 & \epsilon^b y' \\ 0 & \epsilon^b y'^* & 0 \end{pmatrix}$$

Keep on rotating large terms



Fibonacci

Continuing with rotations the error shrinks rapidly.
Take $a = b = 1$ (neutrino oscillation case)

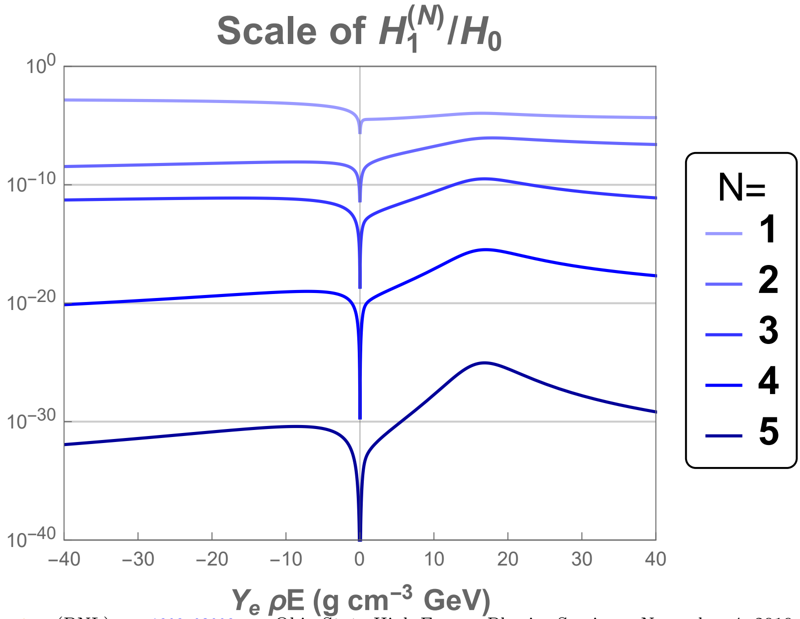


$$F_0 \equiv 0\,, \quad F_1 \equiv 1\,, \quad F_n = F_{n-1} + F_{n-2} \text{ for } n > 1$$

$$\lim_{n\to\infty} F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n$$

X. Zhang, PBD, S. Parke 1909.02009

Exponential (Fibonacci) Improvement



CP violation in matter

The CPV Term in Matter

The amount of CPV is

$$P_{\alpha\beta} - \bar{P}_{\alpha\beta} = \pm 16 J \sin\Delta_{21} \sin\Delta_{31} \sin\Delta_{32} \qquad \alpha \neq \beta$$

where the Jarlskog is

$$J \equiv \Im[U_{\alpha i} U_{\beta j} U^*_{\alpha j} U^*_{\beta i}] \qquad \alpha \neq \beta,\ i \neq j$$

$$J = c_{12} s_{12} c^2_{13} s_{13} c_{23} s_{23} \sin\delta$$

C. Jarlskog PRL 55 (1985)

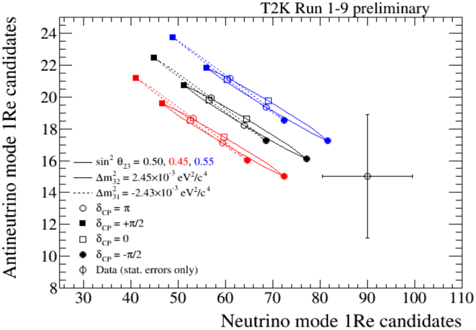
The exact term in matter is known to be

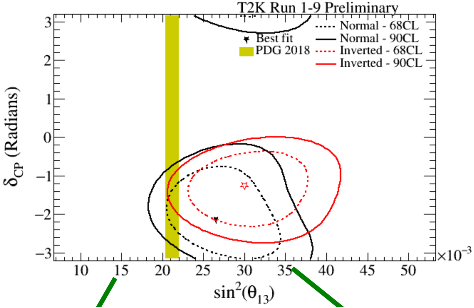
$$\frac{\widehat{J}}{J} = \frac{\Delta m^2_{21} \Delta m^2_{31} \Delta m^2_{32}}{\Delta\widehat{m^2}_{21} \Delta\widehat{m^2}_{31} \Delta\widehat{m^2}_{32}}$$

V. Naumov IJMP 1992

P. Harrison, W. Scott hep-ph/9912435

CPV Tension at T2K





$$J = c_{12}s_{12}c_{13}^2 s_{13}c_{23}s_{23}\sin\delta$$

CPV in Matter

CPV in matter can be written sans $\cos(\frac{1}{3}\cos^{-1}(\cdots))$ term.

$$\frac{\widehat{J}}{J}=\frac{\Delta m^2_{21}\Delta m^2_{31}\Delta m^2_{32}}{\Delta\widehat{m^2}_{21}\Delta\widehat{m^2}_{31}\Delta\widehat{m^2}_{32}}$$

$$\left(\Delta\widehat{m^2}_{21}\Delta\widehat{m^2}_{31}\Delta\widehat{m^2}_{32}\right)^2=(A^2-4B)(B^2-4AC)+(2AB-27C)C$$

$$A\equiv\sum_j\widehat{m^2}_j=\Delta m^2_{31}+\Delta m^2_{21}+a$$

$$B\equiv\sum_{j>k}\widehat{m^2}_j\widehat{m^2}_k=\Delta m^2_{31}\Delta m^2_{21}+a(\Delta m^2_{ee}c^2_{13}+\Delta m^2_{21})$$

$$C\equiv\prod_j\widehat{m^2}_j=a\Delta m^2_{31}\Delta m^2_{21}c^2_{13}c^2_{12}$$

This is the only oscillation quantity in matter that can be written exactly without $\cos(\frac{1}{3}\cos^{-1}(\cdots))$!

H. Yokomakura, K. Kimura, A. Takamura hep-ph/0009141

CPV Factorizes

Thus $\widehat{J}^{-2}$ is fourth order in matter potential:
only two matter corrections are needed.

$$\frac{\widehat{J}}{J}=\frac{1}{|1-(a/\alpha_1)e^{i2\theta_1}||1-(a/\alpha_2)e^{i2\theta_2}|}$$

CPV Factorizes

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$$\frac{\widehat{J}}{J}=\frac{1}{|1-(a/\alpha_1)e^{i2\theta_1}||1-(a/\alpha_2)e^{i2\theta_2}|}$$

CPV in matter can be well approximated:

$$\frac{\widehat{J}}{J}\approx\frac{1}{|1-(a/\Delta m^2_{ee})e^{i2\theta_{13}}||1-(c^2_{13}a/\Delta m^2_{21})e^{i2\theta_{12}}|}$$

PBD, Parke 1902.07185

See also X. Wang, S. Zhou 1901.10882

Precise at the $< 0.04\%$ level!

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See also X. Wang, S. Zhou 1901.10882

Precise at the $< 0.04\%$ level!

CPV Factorizes Part II

- Option 1: Use NHS identity and $\Delta\widehat{m^2}$'s
- Option 2: Use the angles?

$$\frac{\widehat{J}}{J}=\frac{s_{\widehat{12}}c_{\widehat{12}}s_{\widehat{13}}c^2_{\widehat{13}}s_{\widehat{23}}c_{\widehat{23}}\sin\widehat{\delta}}{s_{12}c_{12}s_{13}c^2_{13}s_{23}c_{23}\sin\delta}$$

Toshev: θ_{23}, δ:

S. Toshev MPL A6 (1991) 455

$$\frac{\widehat{J}}{J}=\frac{s_{\widehat{12}}c_{\widehat{12}}s_{\widehat{13}}c^2_{\widehat{13}}}{s_{12}c_{12}s_{13}c^2_{13}}$$

How to split up?

CPV Factorizes Part II

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How to split up?

From DMP:

PBD, H. Minakata, S. Parke 1604.08167

$$\frac{1}{|1-(a/\Delta m^2_{ee})e^{i2\theta_{13}}|}\approx\frac{s_{\widehat{13}}c_{\widehat{13}}}{s_{13}c_{13}}$$

Hopefully:

$$\frac{1}{|1-(c^2_{13}a/\Delta m^2_{21})e^{i2\theta_{12}}|}\approx\frac{s_{\widehat{12}}c_{\widehat{12}}c_{\widehat{13}}}{s_{12}c_{12}c_{13}}$$

CPV Factorizes Part II

- Option 1: Use NHS identity and $\Delta\widehat{m^2}$'s
- Option 2: Use the angles?

$$\frac{\widehat{J}}{J}=\frac{s_{\widehat{12}}c_{\widehat{12}}s_{\widehat{13}}c_{\widehat{13}}^2s_{\widehat{23}}c_{\widehat{23}}\sin\widehat{\delta}}{s_{12}c_{12}s_{13}c_{13}^2s_{23}c_{23}\sin\delta}$$

Toshev: θ_{23}, δ:

S. Toshev MPL A6 (1991) 455

$$\frac{\widehat{J}}{J}=\frac{s_{\widehat{12}}c_{\widehat{12}}s_{\widehat{13}}c_{\widehat{13}}^2}{s_{12}c_{12}s_{13}c_{13}^2}$$

How to split up?

From DMP:

PBD, H. Minakata, S. Parke 1604.08167

$$\frac{1}{|1-(a/\Delta m_{ee}^2)e^{i2\theta_{13}}|}\approx\frac{s_{\widehat{13}}c_{\widehat{13}}}{s_{13}c_{13}}\qquad\sim 0.4\%$$

Hopefully:

$$\frac{1}{|1-(c_{13}^2a/\Delta m_{21}^2)e^{i2\theta_{12}}|}\approx\frac{s_{\widehat{12}}c_{\widehat{12}}c_{\widehat{13}}}{s_{12}c_{12}c_{13}}\qquad\sim 0.4\%$$

Key Cancellation

Expect s_{13}^2 or $\Delta m_{21}^2/\Delta m_{ee}^2 \sim 2-3\%$ precision

The atmospheric term:

$$\Delta m_{31}^2 \quad \Delta m_{ee}^2 \quad \Delta m_{32}^2$$

Solar correction:

$$1 \quad c_{13}^2 \quad \cos 2\theta_{13}$$

Key Cancellation

Expect s_{13}^2 or $\Delta m_{21}^2/\Delta m_{ee}^2 \sim 2-3\%$ precision

The atmospheric term:

Δm_{31}^2	Δm_{ee}^2	Δm_{32}^2
✗	✓	✗

Solar correction:

1	c_{13}^2	$\cos 2\theta_{13}$
✗	✓	✗

Key Cancellation

Expect s_{13}^2 or $\Delta m_{21}^2/\Delta m_{ee}^2 \sim 2-3\%$ precision

The atmospheric term:

$$\Delta m_{31}^2 \quad \Delta m_{ee}^2 \quad \Delta m_{32}^2$$

✗ ✓ ✗

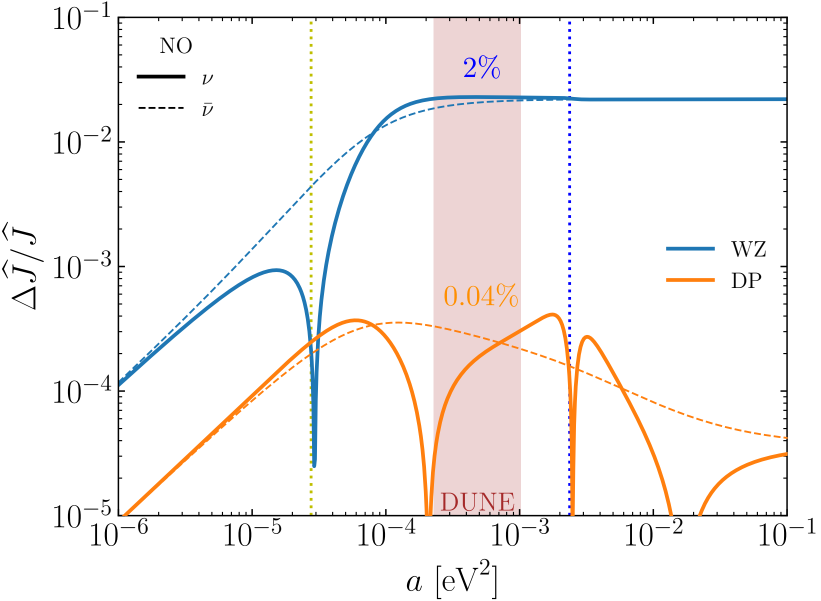
Solar correction:

$$1 \quad c_{13}^2 \quad \cos 2\theta_{13}$$

✗ ✓ ✗

$$\frac{\Delta\widehat{J}}{\widehat{J}} \sim \mathcal{O}\left(s_{13}^2\frac{\Delta m_{21}^2}{\Delta m_{ee}^2}\right) + \mathcal{O}\left[\left(\frac{\Delta m_{21}^2}{\Delta m_{ee}^2}\right)^2\right] \sim 0.04\%$$

CPV In Matter Approximation Precision



New Physics

DUNE and T2HK will unprecedented capabilities to test the three-neutrino oscillation picture

Extend DMP to new physics progress report:

- ☑ Sterile

  S. Parke, X. Zhang 1905.01356

- ☐ NSI
- ☐ Neutrino decay
- ☐ Decoherence
- ☐ …

Given Rosetta, extensions should be considerably simpler

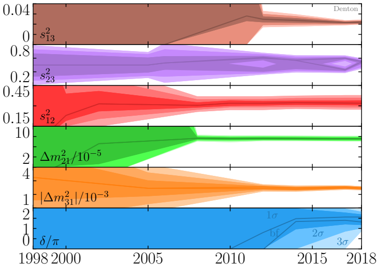
Key Points

- Understanding probabilities in matter is key to current/future LBL
- Long-baseline oscillations are fundamentally three-flavor
- Approximate eigenvalues are key
- Eigenvectors follow from eigenvalues
- Exact and approximate CPV in matter are simpler than expected

Thanks!

Backups

References



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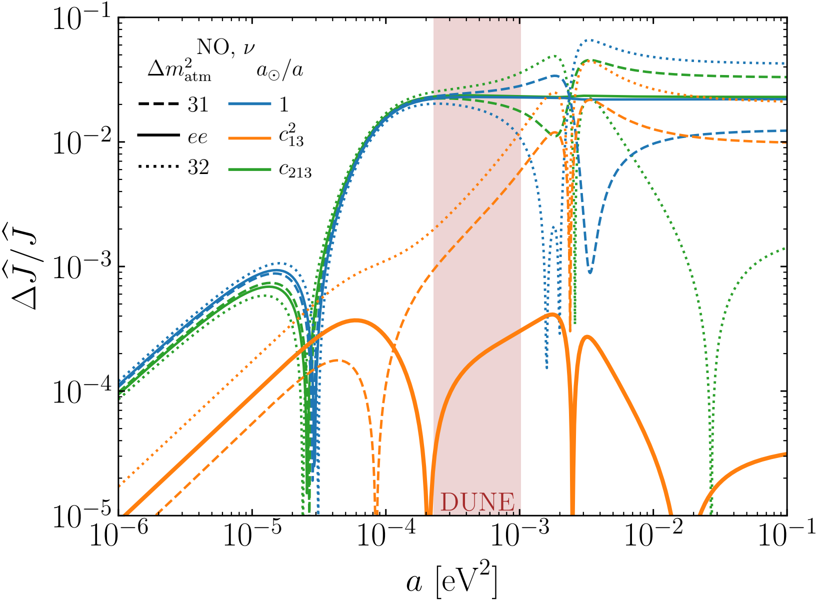
D. Forero, M. Tortola, J. Valle 1205.4018

D. Forero, M. Tortola, J. Valle 1405.7540

P. de Salas, et al. 1708.01186

Factorization Conditions



Proper Expansions

Parameter x is an expansion parameter iff

$$\lim_{x\to 0} P_{\text{approx}}(x) = P_{\text{exact}}(x=0)$$

	ϵ	s_{13}	$a/\Delta m^2_{31}$	
Madrid(like)	✗	✗	✗	Cervera+ hep-ph/0002108
AKT	✓	✓	✓	Agarwalla+ 1302.6773
MP	✓	✗	✗	Minakata, Parke 1505.01826
DMP	✓	✓	✓	PBD+ 1604.08167
AKS	✗	✗	✗	Arafune+ hep-ph/9703351
MF	✓	✗	✗	Freund hep-ph/0103300
AJLOS(48)	✓	✗	✗	Akhmedov+ hep-ph/0402175
AM	✗	✗	✗	Asano, Minakata 1103.4387

$$\epsilon \simeq \frac{\Delta m^2_{21}}{\Delta m^2_{ee}}$$

The Cubic

Math history aside

1. **Ancients** (20-16C BC)
   Babylonians, Greeks, Chinese, Indians, Egyptians:
   thought about cubics, calculated cube roots

$$x^3 = a$$

2. Chinese **Wang Xiaotong** (7C AD):
   numerically solved 25 general cubics
3. Persian **Omar Khayyam** (11C AD):
   realized there are multiple solutions

The Cubic

Math history aside: The Italian Job (16C AD)

4. Scipione del **Ferro**:
   Secret solution, nearly all (didn't know negative numbers)

$$x^3 + mx = n$$

5. Antonio **Fiore**: Ferro's student, from just before his death
6. Niccol **Tartaglia**: Claimed a solution, was challenged by Fiore
7. Gerolamo **Cardano**: Gets Tartaglia's (winner) solution, promises to keep it secret. Later publishes Ferro's solution via Fiore
8. Tartaglia challenges Cardano who denies it. Cardano's student **Ferrari** accepted, Tartaglia lost along with prestige and income

Quartic was (nearly) solved around the same time by Ferrari, *before* the cubic solution was published