

Abstract

Expressing neutrino oscillation probabilities in matter can be done either approximately with very good precision, or exactly with complicated expressions. Exact solutions require solving for the eigenvalues and, in turn, the eigenvectors. While there is no shortcut for the exact expressions for the eigenvalues, given those the eigenvectors can be determined in a straightforward fashion. Finally, I will show that while CPV in matter appears to extremely complicated, it is much simpler than expected both exactly and after approximations.

Recent Results in Neutrino Oscillation Theory

Peter B. Denton

CERN Neutrino Platform

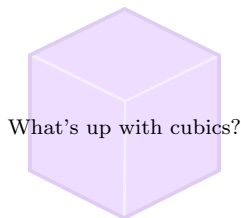
October 8, 2019

BROOKHAVEN
NATIONAL LABORATORY



Path

1. **Statement** of oscillation question
2. Get the eigen**values**
3. Get the eigen**vectors**
4. **CP** violation in matter



Statement of oscillation question

The Big Question

What is $P(\nu_\mu \rightarrow \nu_e)$?

$$P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) = |\mathcal{A}_{\mu e}|^2 \quad \mathcal{A}_{\mu e} = \mathcal{A}_{31} + e^{\pm i\Delta_{32}} \mathcal{A}_{21}$$

$$\mathcal{A}_{31} = 2s_{13}c_{13}s_{23} \sin \Delta_{31}$$

$$\mathcal{A}_{21} = 2s_{12}c_{13}(c_{12}c_{23}e^{i\delta} - s_{12}s_{13}s_{23}) \sin \Delta_{21}$$

$$\Delta_{ij} = \Delta m_{ij}^2 L / 4E$$

...in matter?

Now: NOvA, T2K, MINOS, ...

Upcoming: DUNE, T2HK, ...

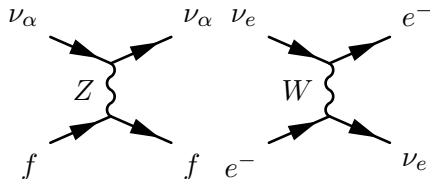
Second maximum: T2HKK? ESSnuSB? ...

Matter Effects Matter

Call Schrödinger equation's eigenvalues m_i^2 and eigenvectors U_i .

$$\mathcal{A}(\nu_\alpha \rightarrow \nu_\beta) = \sum_{i=1}^3 U_{\alpha i}^* U_{\beta i} e^{-im_i^2 L/2E} \quad P = |\mathcal{A}|^2$$

In matter ν 's propagate in a new basis that depends on $a \propto \rho E$.



L. Wolfenstein, [PRD 17 \(1978\)](#)

Eigenvalues: $m_i^2 \rightarrow \widehat{m_i^2}(a)$

Eigenvectors are given by $\theta_{ij} \rightarrow \widehat{\theta_{ij}}(a) \quad \Leftarrow \quad \text{Unitarity}$

Hamiltonian Dynamics

$$H = \frac{1}{2E} \left[U \begin{pmatrix} 0 & & \\ & \Delta m_{21}^2 & \\ & & \Delta m_{31}^2 \end{pmatrix} U^\dagger + \begin{pmatrix} a & & \\ & 0 & \\ & & 0 \end{pmatrix} \right]$$

$$a = 2\sqrt{2}G_F N_e E$$

$$U = \begin{pmatrix} 1 & & \\ & c_{23} & s_{23} \\ & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & & s_{13}e^{-i\delta} \\ & 1 & \\ -s_{13}e^{i\delta} & & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & \\ -s_{12} & c_{12} & \\ & & 1 \end{pmatrix}$$

Find eigenvalues and eigenvectors:

$$H = \frac{1}{2E} \hat{U} \begin{pmatrix} 0 & & \\ & \widehat{\Delta m}_{21}^2 & \\ & & \widehat{\Delta m}_{31}^2 \end{pmatrix} \hat{U}^\dagger$$

J. Kopp [physics/0610206](#)

Computationally works, but we can do better than a **black box**...
Analytic expression?

Get the eigen**values**

Eigenvalues Analytically: The Exact Solution

Solve the cubic characteristic equation: eigen**values**

G. Cardano *Ars Magna* 1545

V. Barger, et al. [PRD 22 \(1980\) 2718](#)

H. Zaglauer, K. Schwarzer [Z.Phys. C40 \(1988\) 273](#)

“Unfortunately, the algebra is rather impenetrable.”

V. Barger, et al.

Then write down eigen**vectors** (mixing angles)

H. Zaglauer, K. Schwarzer [Z.Phys. C40 \(1988\) 273](#)

K. Kimura, A. Takamura, H. Yokomakura [hep-ph/0205295](#)

[PBD](#), S. Parke, X. Zhang [1907.02534](#)

The Cubic

Math history aside

Linear: $ax + b = 0$



Quadratic: $ax^2 + bx + c = 0$



Cubic: $ax^3 + bx^2 + cx + d = 0$



Quartic: $ax^4 + bx^3 + cx^2 + dx + e = 0$



Qunitic+: $ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$



Abel-Ruffini theorem, 1824

Eigenvalues Analytically: The Exact Solution

The cubic solution (in neutrino terms)

$$\widehat{m^2_1} = \frac{A}{3} - \frac{1}{3}\sqrt{A^2 - 3BS} - \frac{\sqrt{3}}{3}\sqrt{A^2 - 3B}\sqrt{1 - S^2}$$

$$\widehat{m^2_2} = \frac{A}{3} - \frac{1}{3}\sqrt{A^2 - 3BS} + \frac{\sqrt{3}}{3}\sqrt{A^2 - 3B}\sqrt{1 - S^2}$$

$$\widehat{m^2_3} = \frac{A}{3} + \frac{2}{3}\sqrt{A^2 - 3BS}$$

$$A = \Delta m_{21}^2 + \Delta m_{31}^2 + a$$

$$B = \Delta m_{21}^2 \Delta m_{31}^2 + a [c_{13}^2 \Delta m_{31}^2 + (c_{12}^2 c_{13}^2 + s_{13}^2) \Delta m_{21}^2]$$

$$C = a \Delta m_{21}^2 \Delta m_{31}^2 c_{12}^2 c_{13}^2$$

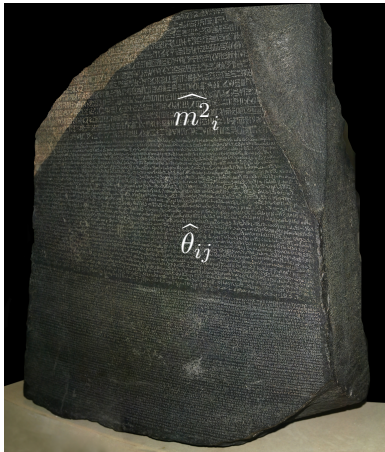
$$S = \cos \left\{ \frac{1}{3} \cos^{-1} \left[\frac{2A^3 - 9AB + 27C}{2(A^2 - 3B)^{3/2}} \right] \right\}$$

Get the eigenvectors

Exact Neutrino Oscillations in Matter: Mixing Angles

$$\begin{aligned}
 s_{12}^2 &= \frac{-\left[(\widehat{m^2_2})^2 - \alpha\widehat{m^2_2} + \beta\right] \Delta\widehat{m^2_{31}}}{\left[(\widehat{m^2_1})^2 - \alpha\widehat{m^2_1} + \beta\right] \Delta\widehat{m^2_{32}} - \left[(\widehat{m^2_2})^2 - \alpha\widehat{m^2_2} + \beta\right] \Delta\widehat{m^2_{31}}} \\
 s_{13}^2 &= \frac{(\widehat{m^2_3})^2 - \alpha\widehat{m^2_3} + \beta}{\Delta\widehat{m^2_{31}} \Delta\widehat{m^2_{32}}} \\
 s_{23}^2 &= \frac{s_{23}^2 E^2 + c_{23}^2 F^2 + 2c_{23}s_{23}c_\delta EF}{E^2 + F^2} \\
 e^{-i\delta} &= \frac{c_{23}s_{23} (e^{-i\delta} E^2 - e^{i\delta} F^2) + (c_{23}^2 - s_{23}^2) EF}{\sqrt{(s_{23}^2 E^2 + c_{23}^2 F^2 + 2EFc_{23}s_{23}c_\delta) (c_{23}^2 E^2 + s_{23}^2 F^2 - 2EFc_{23}s_{23}c_\delta)}} \\
 \alpha &= c_{13}^2 \Delta m_{31}^2 + (c_{12}^2 c_{13}^2 + s_{13}^2) \Delta m_{21}^2, \quad \beta = c_{12}^2 c_{13}^2 \Delta m_{21}^2 \Delta m_{31}^2 \\
 E &= c_{13}s_{13} \left[\left(\widehat{m^2_3} - \Delta m_{21}^2 \right) \Delta m_{31}^2 - s_{12}^2 \left(\widehat{m^2_3} - \Delta m_{31}^2 \right) \Delta m_{21}^2 \right] \\
 F &= c_{12}s_{12}c_{13} \left(\widehat{m^2_3} - \Delta m_{31}^2 \right) \Delta m_{21}^2
 \end{aligned}$$

H. Zaglauer, K. Schwarzer, [Z.Phys. C40 \(1988\) 273](#)



Eigenvalues to Eigenvectors

KTY pushed calculating the eigenvectors from the eigenvalues.

K. Kimura, A. Takamura, H. Yokomakura [hep-ph/0205295](#)

Expressions for:

$$\hat{U}_{\alpha i} \hat{U}_{\beta i}^*$$

for $\alpha \neq \beta$.

Wanted to preserve phase information for $\hat{\delta}$.

Eigenvalues: the Rosetta Stone

We realized:

$$|\widehat{U}_{\alpha i}|^2 = \frac{(\widehat{m^2}_i - \xi_\alpha)(\widehat{m^2}_i - \chi_\alpha)}{\Delta\widehat{m^2}_{ij}\Delta\widehat{m^2}_{ik}}$$

PBD, S. Parke, X. Zhang [1907.02534](#)

where ξ_α and χ_α are the submatrix eigenvalues of H_α

$$H = \begin{pmatrix} H_{\alpha\alpha} & H_{\alpha\beta} & H_{\alpha\gamma} \\ H_{\beta\alpha} & H_{\beta\beta} & H_{\beta\gamma} \\ H_{\gamma\alpha} & H_{\gamma\beta} & H_{\gamma\gamma} \end{pmatrix} \rightarrow H_\alpha = \begin{pmatrix} H_{\beta\beta} & H_{\beta\gamma} \\ H_{\gamma\beta} & H_{\gamma\gamma} \end{pmatrix}$$

e.g.

$$\xi_e + \chi_e = \Delta m_{21}^2 + \Delta m_{ee}^2 c_{13}^2$$

$$\xi_e \chi_e = \Delta m_{21}^2 [\Delta m_{ee}^2 c_{13}^2 c_{12}^2 + \Delta m_{21}^2 (s_{12}^2 c_{12}^2 - s_{13}^2 s_{12}^2 c_{12}^2)]$$

$$\Delta m_{ee}^2 = c_{12}^2 \Delta m_{31}^2 + s_{12}^2 \Delta m_{32}^2$$

H. Nunokawa, S. Parke, R. Z. Funchal, [hep-ph/0503283](#)

Eigenvalues: the Rosetta Stone

$$s_{13}^2 = |\widehat{U}_{e3}|^2 = \frac{(\widehat{m}_3^2 - \xi_e)(\widehat{m}_3^2 - \chi_e)}{\Delta\widehat{m}_{31}^2\Delta\widehat{m}_{32}^2}$$

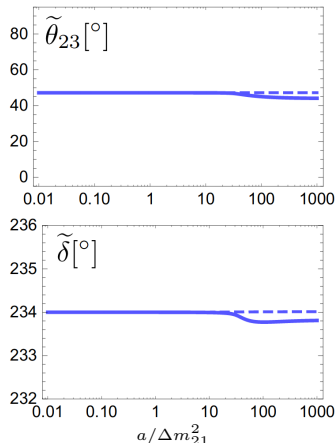
$$s_{12}^2 c_{13}^2 = |\widehat{U}_{e2}|^2 = -\frac{(\widehat{m}_2^2 - \xi_e)(\widehat{m}_2^2 - \chi_e)}{\Delta\widehat{m}_{32}^2\Delta\widehat{m}_{21}^2}$$

$$s_{23}^2 c_{13}^2 = |\widehat{U}_{\mu 3}|^2 = \frac{(\widehat{m}_3^2 - \xi_\mu)(\widehat{m}_3^2 - \chi_\mu)}{\Delta\widehat{m}_{31}^2\Delta\widehat{m}_{32}^2}$$

What about $\widehat{\delta}$?

CPV From Rosetta

$\hat{\delta}$ nearly constant, but have to get it right



Toshev identity:

$$\sin \hat{\delta} = \frac{\sin 2\theta_{23}}{\sin 2\hat{\theta}_{23}} \sin \delta$$

Get the sign of $\cos \hat{\delta}$ from e.g. $|\hat{U}_{\mu 1}|^2$.

S. Toshev [MPL A6 \(1991\) 455](#)

Z-z. Xing, S. Zhou
Y-L. Zhou, [1802.00990](#)

EIGENVECTORS FROM EIGENVALUES

PETER B. DENTON, STEPHEN J. PARKE, TERENCE TAO, AND XINING ZHANG

ABSTRACT. We present a new method of succinctly determining eigenvectors from eigenvalues. Specifically, we relate the norm squared of the elements of eigenvectors to the eigenvalues and the submatrix eigenvalues.

$$|v_{i,j}|^2 = \frac{\prod_{k=1}^{n-1} (\lambda_i - \xi_{j,k})}{\prod_{k=1; k \neq i}^n (\lambda_i - \lambda_k)}$$

Proofs

1. From previous result with $n - 1$ subvectors using derivatives
L. Erdos, B. Schlein, H-T. Yau, [0711.1730](#)
T. Tao, V. Vu, [0906.0510](#)
2. Geometric formulation with exterior algebra
3. Using determinants and a Cauchy-Binet variant
4. Adjugate matrices
Can get off-diagonal elements, thus CP phase
5. Cramer's rule
6. Two other mathematicians provided other proofs
7. Another mathematician generalized it to all square matrices

[https://terrytao.wordpress.com/2019/08/13/
eigenvectors-from-eigenvalues/](https://terrytao.wordpress.com/2019/08/13/eigenvectors-from-eigenvalues/)

CP violation in matter

The CPV Term in Matter

The amount of CPV is

$$P_{\alpha\beta} - \bar{P}_{\alpha\beta} = \pm 16J \sin \Delta_{21} \sin \Delta_{31} \sin \Delta_{32} \quad \alpha \neq \beta$$

where the Jarlskog is

$$J \equiv \Im[U_{\alpha i} U_{\beta j} U_{\alpha j}^* U_{\beta i}^*] \quad \alpha \neq \beta, i \neq j$$

$$J = c_{12}s_{12}c_{13}^2s_{13}c_{23}s_{23} \sin \delta$$

C. Jarlskog, [PRL 55 \(1985\)](#)

The exact term in matter is known to be

$$\frac{\hat{J}}{J} = \frac{\Delta m_{21}^2 \Delta m_{31}^2 \Delta m_{32}^2}{\widehat{\Delta m_{21}^2} \widehat{\Delta m_{31}^2} \widehat{\Delta m_{32}^2}}$$

V. Naumov, [IJMP 1992](#)

P. Harrison, W. Scott, [hep-ph/9912435](#)

CPV in Matter

CPV in matter can be written sans $\cos(\frac{1}{3}\cos^{-1}(\dots))$ term.

$$\frac{\widehat{J}}{J} = \frac{\Delta m_{21}^2 \Delta m_{31}^2 \Delta m_{32}^2}{\widehat{\Delta m}_{21}^2 \widehat{\Delta m}_{31}^2 \widehat{\Delta m}_{32}^2}$$

$$\left(\widehat{\Delta m}_{21}^2 \widehat{\Delta m}_{31}^2 \widehat{\Delta m}_{32}^2\right)^2 = (A^2 - 4B)(B^2 - 4AC) + (2AB - 27C)C$$

$$A \equiv \sum_j \widehat{m}_{2j}^2 = \Delta m_{31}^2 + \Delta m_{21}^2 + a$$

$$B \equiv \sum_{j>k} \widehat{m}_{2j}^2 \widehat{m}_{2k}^2 = \Delta m_{31}^2 \Delta m_{21}^2 + a(\Delta m_{ee}^2 c_{13}^2 + \Delta m_{21}^2)$$

$$C \equiv \prod_j \widehat{m}_{2j}^2 = a \Delta m_{31}^2 \Delta m_{21}^2 c_{13}^2 c_{12}^2$$

This is the only oscillation quantity in matter that can be written exactly without $\cos(\frac{1}{3}\cos^{-1}(\dots))$!

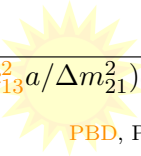
H. Yokomakura, K. Kimura, A. Takamura, [hep-ph/0009141](#)

CPV Factorizes

Thus $\hat{J}^{-2}$ is fourth order in matter potential:
only two matter corrections are needed.

$$\frac{\hat{J}}{J} = \frac{1}{|1 - (a/\alpha_1)e^{i2\theta_1}| |1 - (a/\alpha_2)e^{i2\theta_2}|}$$

CPV in matter can be well approximated:


$$\frac{\hat{J}}{J} \approx \frac{1}{|1 - (a/\Delta m_{ee}^2)e^{i2\theta_{13}}| |1 - (c_{13}^2 a/\Delta m_{21}^2)e^{i2\theta_{12}}|}$$

PBD, Parke, [1902.07185](#)

See also X. Wang, S. Zhou, [1901.10882](#)

Precise at the $< 0.04\%$ level!

Key Cancellation

Expect s_{13}^2 or $\Delta m_{21}^2/\Delta m_{ee}^2 \sim 2 - 3\%$ precision

The atmospheric term:

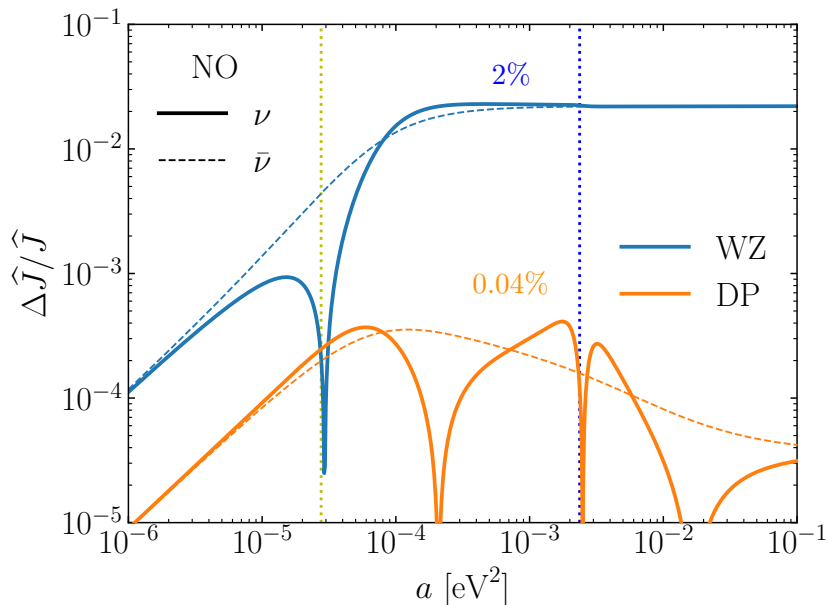
$$\begin{array}{ccc} \Delta m_{31}^2 & \Delta m_{ee}^2 & \Delta m_{32}^2 \\ \text{X} & \checkmark & \text{X} \end{array}$$

Solar correction:

$$\begin{array}{ccc} 1 & c_{13}^2 & \cos 2\theta_{13} \\ \text{X} & \checkmark & \text{X} \end{array}$$

$$\frac{\Delta \hat{J}}{\hat{J}} \sim \mathcal{O} \left(s_{13}^2 \frac{\Delta m_{21}^2}{\Delta m_{ee}^2} \right) + \mathcal{O} \left[\left(\frac{\Delta m_{21}^2}{\Delta m_{ee}^2} \right)^2 \right] \sim 0.04\%$$

CPV In Matter Approximation Precision



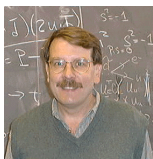
Key Points

- ▶ Understanding probabilities in matter is key to current/future LBL
- ▶ Long-baseline oscillations are fundamentally three-flavor
- ▶ Eigenvectors follow from eigenvalues
- ▶ Exact and approximate CPV in matter are simpler than expected

Questions!

Backups

Analytic Oscillation Probability Collaborators



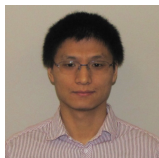
Stephen Parke



Hisakazu Minakata



Gabriela Barenboim



Xining Zhang



Christoph Ternes



Terry Tao

1604.08167, 1806.01277, 1808.09453,
1902.00517, 1902.07185, 1907.02534
1908.03795, 1909.02009

github.com/PeterDenton/Nu-Pert
github.com/PeterDenton/Nu-Pert-Compare

Analytic Oscillation Probabilities in Matter

☑ Solar: $P_{ee} \simeq \sin^2 \theta_\odot$

Approx: S. Mikheev, A. Smirnov, [Nuovo Cim. C9 \(1986\) 17-26](#)

Exact: S. Parke, [PRL 57 \(1986\) 2322](#)

☑ Long-baseline: All three flavors

Exact: H. Zaglauer, K. Schwarzer, [Z.Phys. C40 \(1988\) 273](#)

Approx: [PBD](#), H. Minakata, S. Parke, [1604.08167](#)

☑ ν_e disappearance (neutrino factory):

$$\widehat{\Delta m^2_{ee}} = \widehat{m^2_3} - (\widehat{m^2_1} + \widehat{m^2_2} - \Delta m^2_{21} c^2_{12})$$

[PBD](#), S. Parke, [1808.09453](#)

☐ Atmospheric

Values and Vectors

Probability amplitude:

$$\mathcal{A}_{\alpha\beta} = \sum_i \hat{U}_{\alpha i}^* \hat{U}_{\beta i} e^{-i\hat{m}_i^2 L/2E}$$

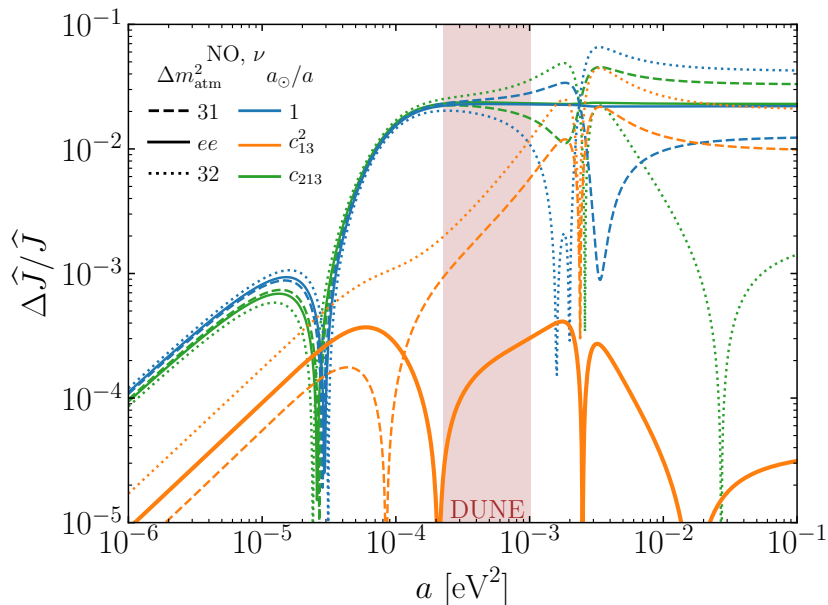
- **Eigenvalues** give the frequencies of the oscillations

Where should DUNE be?

- **Eigenvectors** give the amplitudes of the oscillations

How many events will DUNE see?

Factorization Conditions



The Cubic

Math history aside

1. **Ancients** (20-16C BC)

Babylonians, Greeks, Chinese, Indians, Egyptians:
thought about cubics, calculated cube roots

$$x^3 = a$$

2. Chinese **Wang Xiaotong** (7C AD):

numerically solved 25 general cubics

3. Persian **Omar Khayyam** (11C AD):

realized there are multiple solutions

The Cubic

Math history aside: The Italian Job (16C AD)

4. Scipione del **Ferro**:

Secret solution, nearly all (didn't know negative numbers)

$$x^3 + mx = n$$

5. Antonio **Fiore**: Ferro's student, from just before his death

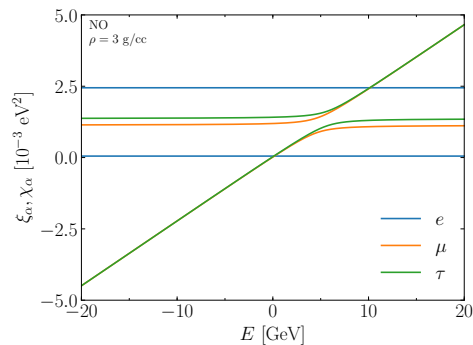
6. Niccol **Tartaglia**: Claimed a solution, was challenged by Fiore

7. Gerolamo **Cardano**: Gets Tartaglia's (winner) solution, promises to keep it secret. Later publishes Ferro's solution via Fiore

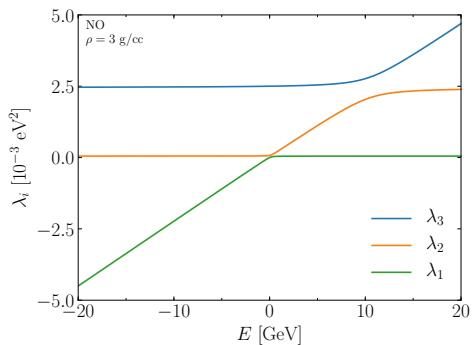
8. Tartaglia challenges Cardano who denies it. Cardano's student **Ferrari** accepted, Tartaglia lost along with prestige and income

Quartic was (nearly) solved around the same time by Ferrari,
before the cubic solution was published

Submatrix Eigenvalues



Submatrix Eigenvalues



Eigenvalues

In General

Two flavor:

$$|\widehat{U}_{\alpha i}|^2 = \frac{\widehat{m^2_i} - \xi_\alpha}{\Delta \widehat{m^2_{ij}}}$$

leads to

$$\begin{aligned} \sin^2 \widehat{\theta} &= |\widehat{U}_{e2}|^2 = \frac{\widehat{m^2_2} - \xi_e}{\widehat{m^2_2} - \widehat{m^2_1}} \\ &= \frac{1}{2} \left(1 - \frac{\Delta m^2 \cos 2\theta - a}{\sqrt{(\Delta m^2 \cos 2\theta - a)^2 + (\Delta m^2 \sin 2\theta)^2}} \right) \end{aligned}$$

Numerically checked for $N = 4, 5$.

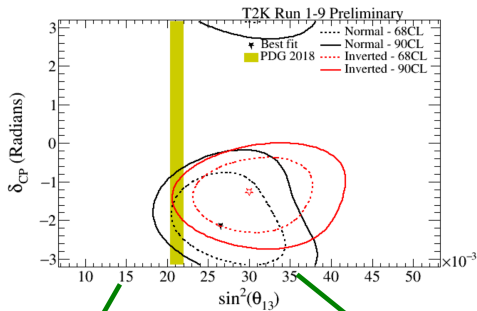
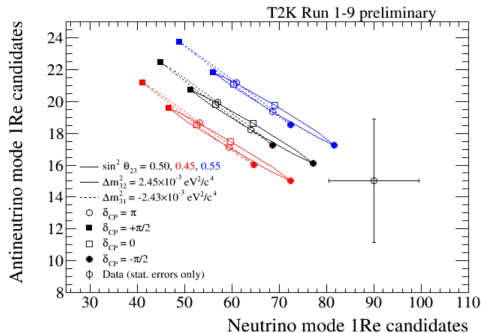
True for all N ?

A Cheery Firehose

1. Terry posted on the same question with a different answer
2. We emailed our result, < 2 hours later:
 - ▶ “Very nice identity!”
 - ▶ New result
 - ▶ 3 distinct proofs
3. 6d later, we’ve sorted 1 proof, send a draft, < 1 hr later:
 - ▶ Agrees to a paper
 - ▶ Adds a corollary
 - ▶ Adds several new observation
4. Barely processed that, another email $< \frac{1}{2}$ day later
 - ▶ A more general proof
5. We sent confirmation that the v_i are normed
6. < 1 day showed how that followed from proof 4

“He’s famously like a cheery firehose of mathematics
Guess he’s power-washing you today”

CPV Tension at T2K



$$J = c_{12}s_{12}c_{13}^2s_{13}c_{23}s_{23} \sin \delta$$

CPV Factorizes Part II

- ▶ Option 1: Use NHS identity and $\widehat{\Delta m^2}$'s
- ▶ Option 2: Use the angles?

$$\frac{\widehat{J}}{J} = \frac{s_{12}\widehat{c_{12}}s_{13}\widehat{c_{13}^2}s_{23}\widehat{c_{23}}\sin\widehat{\delta}}{s_{12}c_{12}s_{13}c_{13}^2s_{23}c_{23}\sin\delta}$$

Toshev: θ_{23}, δ :

S. Toshev [MPL A6 \(1991\) 455](#)

$$\frac{\widehat{J}}{J} = \frac{s_{12}\widehat{c_{12}}s_{13}\widehat{c_{13}^2}}{s_{12}c_{12}s_{13}c_{13}^2}$$

How to split up?

From DMP:

[PBD](#), H. Minakata, S. Parke, [1604.08167](#)

$$\frac{1}{|1 - (a/\Delta m_{ee}^2)e^{i2\theta_{13}}|} \approx \frac{s_{13}\widehat{c_{13}}}{s_{13}c_{13}} \sim 0.4\%$$

Hopefully:

$$\frac{1}{|1 - (c_{13}^2 a/\Delta m_{21}^2)e^{i2\theta_{12}}|} \approx \frac{s_{12}\widehat{c_{12}}\widehat{c_{13}}}{s_{12}c_{12}c_{13}} \sim 0.4\%$$