Abstract

Expressing neutrino oscillation probabilities in matter can be done either approximately with very good precision, or exactly with complicated expressions. Exact solutions require solving for the eigenvalues and, in turn, the eigenvectors. While there is no shortcut for the exact expressions for the eigenvalues, given those the eigenvectors can be determined in a straightforward fashion.

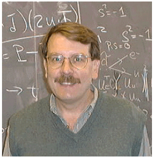
Exact Neutrino Oscillation Probabilities in Matter

Peter B. Denton

TomFest

August 14, 2019

Analytic Oscillation Probability Collaborators



Stephen Parke

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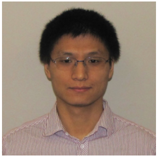


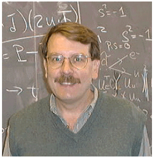
Xining Zhang

Christoph Ternes

1604.08167, 1806.01277, 1808.09453,
1902.00517, 1902.07185, 1907.02534

github.com/PeterDenton/Nu-Pert
github.com/PeterDenton/Nu-Pert-Compare

Analytic Oscillation Probability Collaborators



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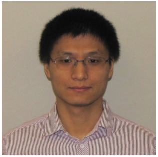

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Christoph Ternes

Terry Tao

1604.08167, 1806.01277, 1808.09453,
1902.00517, 1902.07185, 1907.02534
1908.03795
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What is $P(\nu_\mu \to \nu_e)$?

$$P(\overset{(-)}{\nu}_\mu \to \overset{(-)}{\nu}_e) = |\mathcal{A}_{\mu e}|^2 \qquad \mathcal{A}_{\mu e} = \mathcal{A}_{31} + e^{\pm i\Delta_{32}}\mathcal{A}_{21}$$

$$\mathcal{A}_{31} = 2s_{13}c_{13}s_{23}\sin\Delta_{31}$$

$$\mathcal{A}_{21} = 2s_{12}c_{13}(c_{12}c_{23}e^{i\delta} - s_{12}s_{13}s_{23})\sin\Delta_{21}$$

$$\Delta_{ij} = \Delta m^2{}_{ij}L/4E$$

The Billion Dollar Question

What is $P(\nu_\mu \to \nu_e)$?

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$$\Delta_{ij} = \Delta m^2{}_{ij}L/4E$$

...in matter?

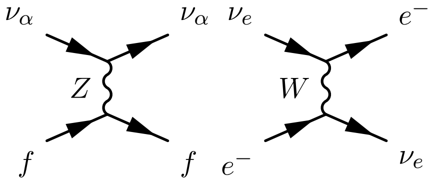
Now: NOvA, T2K, MINOS, ...
Upcoming: DUNE, T2HK, ...
Second maximum: T2HKK? ESSnuSB? ...

Matter Effects Matter

Call Schrödinger equation's eigenvalues m_i^2 and eigenvectors U_i.

$$\mathcal{A}(\nu_\alpha \to \nu_\beta) = \sum_{i=1}^{3} U_{\alpha i}^* U_{\beta i} e^{-i m_i^2 L/2E} \qquad P = |\mathcal{A}|^2$$

In matter ν's propagate in a new basis that depends on $a \propto \rho E$.



L. Wolfenstein, PRD 17 (1978)

Eigenvalues: $m_i^2 \to \widehat{m^2}_i(a)$

Eigenvectors are given by $\theta_{ij} \to \widehat{\theta}_{ij}(a) \qquad \Leftarrow \qquad$ Unitarity

Hamiltonian Dynamics

$$H = \frac{1}{2E}\left[U\begin{pmatrix}0 & & \\ & \Delta m^2_{21} & \\ & & \Delta m^2_{31}\end{pmatrix}U^\dagger + \begin{pmatrix}a & & \\ & 0 & \\ & & 0\end{pmatrix}\right]$$

$a = 2\sqrt{2}G_F N_e E$

$$U = \begin{pmatrix}1 & & \\ & c_{23} & s_{23} \\ & -s_{23} & c_{23}\end{pmatrix}\begin{pmatrix}c_{13} & & s_{13}e^{-i\delta} \\ & 1 & \\ -s_{13}e^{i\delta} & & c_{13}\end{pmatrix}\begin{pmatrix}c_{12} & s_{12} & \\ -s_{12} & c_{12} & \\ & & 1\end{pmatrix}$$

Find eigenvalues and eigenvectors:

$$H = \frac{1}{2E}\widehat{U}\begin{pmatrix}0 & & \\ & \Delta\widehat{m^2}_{21} & \\ & & \Delta\widehat{m^2}_{31}\end{pmatrix}\widehat{U}^\dagger$$

Analytic expression?

Eigenvalues Analytically: The Exact Solution

Solve the cubic characteristic equation

G. Cardano Ars Magna 1545

V. Barger, et al., PRD 22 (1980) 2718

H. Zaglauer, K. Schwarzer, Z.Phys. C40 (1988) 273

Then write down eigenvectors (mixing angles)

H. Zaglauer, K. Schwarzer, Z.Phys. C40 (1988) 273

K. Kimura, A. Takamura, H. Yokomakura hep-ph/0205295

PBD, S. Parke, X. Zhang 1907.02534

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"Unfortunately, the algebra is rather impenetrable."

V. Barger, et al.

Eigenvalues Analytically: The Exact Solution

Solve the cubic characteristic equation.

$$\widehat{m^2}_1 = \frac{A}{3} - \frac{1}{3}\sqrt{A^2-3B}S - \frac{\sqrt{3}}{3}\sqrt{A^2-3B}\sqrt{1-S^2}$$
$$\widehat{m^2}_2 = \frac{A}{3} - \frac{1}{3}\sqrt{A^2-3B}S + \frac{\sqrt{3}}{3}\sqrt{A^2-3B}\sqrt{1-S^2}$$
$$\widehat{m^2}_3 = \frac{A}{3} + \frac{2}{3}\sqrt{A^2-3B}S$$
$$A = \Delta m^2_{21} + \Delta m^2_{31} + a$$
$$B = \Delta m^2_{21}\Delta m^2_{31} + a\left[c^2_{13}\Delta m^2_{31} + (c^2_{12}c^2_{13} + s^2_{13})\Delta m^2_{21}\right]$$
$$C = a\Delta m^2_{21}\Delta m^2_{31}c^2_{12}c^2_{13}$$
$$S = \cos\left\{\frac{1}{3}\cos^{-1}\left[\frac{2A^3 - 9AB + 27C}{2\left(A^2-3B\right)^{3/2}}\right]\right\}$$

Exact Neutrino Oscillations in Matter: Mixing Angles

$$s^2_{\widehat{12}} = \frac{-\left[(\widehat{m^2}_2)^2 - \alpha\widehat{m^2}_2 + \beta\right]\Delta\widehat{m^2}_{31}}{\left[(\widehat{m^2}_1)^2 - \alpha\widehat{m^2}_1 + \beta\right]\Delta\widehat{m^2}_{32} - \left[(\widehat{m^2}_2)^2 - \alpha\widehat{m^2}_2 + \beta\right]\Delta\widehat{m^2}_{31}}$$

$$s^2_{\widehat{13}} = \frac{(\widehat{m^2}_3)^2 - \alpha\widehat{m^2}_3 + \beta}{\Delta\widehat{m^2}_{31}\Delta\widehat{m^2}_{32}}$$

$$s^2_{\widehat{23}} = \frac{s^2_{23}E^2 + c^2_{23}F^2 + 2c_{23}s_{23}c_\delta EF}{E^2 + F^2}$$

$$e^{-i\widehat{\delta}} = \frac{c_{23}s_{23}\left(e^{-i\delta}E^2 - e^{i\delta}F^2\right) + \left(c^2_{23} - s^2_{23}\right)EF}{\sqrt{\left(s^2_{23}E^2 + c^2_{23}F^2 + 2EFc_{23}s_{23}c_\delta\right)\left(c^2_{23}E^2 + s^2_{23}F^2 - 2EFc_{23}s_{23}c_\delta\right)}}$$

$$\alpha = c^2_{13}\Delta m^2_{31} + \left(c^2_{12}c^2_{13} + s^2_{13}\right)\Delta m^2_{21}\,, \ \ \beta = c^2_{12}c^2_{13}\Delta m^2_{21}\Delta m^2_{31}$$

$$E = c_{13}s_{13}\left[\left(\widehat{m^2}_3 - \Delta m^2_{21}\right)\Delta m^2_{31} - s^2_{12}\left(\widehat{m^2}_3 - \Delta m^2_{31}\right)\Delta m^2_{21}\right]$$

$$F = c_{12}s_{12}c_{13}\left(\widehat{m^2}_3 - \Delta m^2_{31}\right)\Delta m^2_{21}$$

H. Zaglauer, K. Schwarzer, Z.Phys. C40 (1988) 273

Too Hard: Approximations

- Small matter potential: $a/\Delta m^2$

Y. Li, Y. Wang, Z-z. Xing, 1605.00900

- s_{13}, s_{13}^2

A. Cervera, et al., hep-ph/0002108

H. Minakata, 0910.5545

K. Asano, H. Minakata, 1103.4387

- $\Delta m_{21}^2/\Delta m_{31}^2 \sim 0.03$

J. Arafune, J. Sato, hep-ph/9607437

A. Cervera, et al., hep-ph/0002108

M. Freund, hep-ph/0103300

E. Akhmedov, et al., hep-ph/0402175

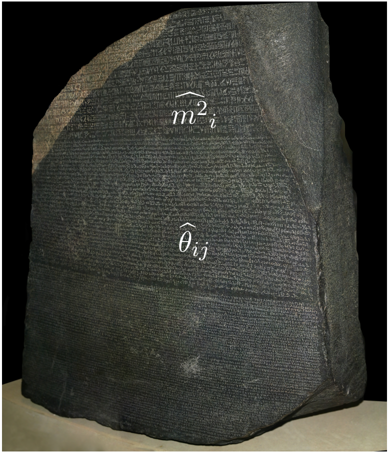
S. Agarwalla, Y. Kao, T. Takeuchi, 1302.6773

M. Blennow, A. Smirnov, 1306.2903

H. Minakata, S. Parke, 1505.01826

PBD, H. Minakata, S. Parke, 1604.08167

(See G. Barenboim, PBD, S. Parke, C. Ternes 1902.00517 for a review)


$\widehat{m^2}_i$
$\widehat{\theta}_{ij}$

Eigenvalues to Eigenvectors

KTY pushed calculating the eigenvectors from the eigenvalues.

K. Kimura, A. Takamura, H. Yokomakura hep-ph/0205295

Expressions for:

$$\widehat{U}_{\alpha i}\widehat{U}^*_{\beta i}$$

for $\alpha \neq \beta$.
Wanted to preserve phase information for $\widehat{\delta}$.

Eigenvalues: the Rosetta Stone

We realized:

$$|\widehat{U}_{\alpha i}|^2 = \frac{(\widehat{m^2}_i - \xi_\alpha)(\widehat{m^2}_i - \chi_\alpha)}{\Delta\widehat{m^2}_{ij}\Delta\widehat{m^2}_{ik}}$$

where ξ_α and χ_α are the submatrix eigenvalues of

$$H_\alpha \equiv \begin{pmatrix} H_{\beta\beta} & H_{\beta\gamma} \\ H_{\gamma\beta} & H_{\gamma\gamma} \end{pmatrix}$$

e.g.

$$\xi_e + \chi_e = \Delta m^2_{21} + \Delta m^2_{ee} c^2_{13}$$

$$\xi_e \chi_e = \Delta m^2_{21}[\Delta m^2_{ee} c^2_{13} c^2_{12} + \Delta m^2_{21}(s^2_{12} c^2_{12} - s^2_{13} s^2_{12} c^2_{12})]$$

Eigenvalues: the Rosetta Stone

$$s^2_{\widehat{13}} = |\widehat{U}_{e3}|^2 = \frac{(\widehat{m^2}_3 - \xi_e)(\widehat{m^2}_3 - \chi_e)}{\Delta\widehat{m^2}_{31}\Delta\widehat{m^2}_{32}}$$

$$s^2_{\widehat{12}}c^2_{\widehat{13}} = |\widehat{U}_{e2}|^2 = -\frac{(\widehat{m^2}_2 - \xi_e)(\widehat{m^2}_2 - \chi_e)}{\Delta\widehat{m^2}_{32}\Delta\widehat{m^2}_{21}}$$

$$s^2_{\widehat{23}}c^2_{\widehat{13}} = |\widehat{U}_{\mu 3}|^2 = \frac{(\widehat{m^2}_3 - \xi_\mu)(\widehat{m^2}_3 - \chi_\mu)}{\Delta\widehat{m^2}_{31}\Delta\widehat{m^2}_{32}}$$

What about $\widehat{\delta}$?

CPV From Rosetta

Toshev identity:

$$\sin\widehat{\delta} = \frac{\sin 2\theta_{23}}{\sin 2\widehat{\theta}_{23}}\sin\delta$$

S. Toshev MPL A6 (1991) 455

Get the sign of $\cos\widehat{\delta}$ from e.g. $|U_{\mu 1}|^2$.

In General

Two flavor:

$$|\widehat{U}_{\alpha i}|^2 = \frac{\widehat{m^2}_i - \xi_\alpha}{\Delta\widehat{m^2}_{ij}}$$

leads to

$$\sin^2\widehat{\theta} = |\widehat{U}_{e2}|^2 = \frac{\widehat{m^2}_2 - \xi_e}{\widehat{m^2}_2 - \widehat{m^2}_1}$$

$$= \frac{1}{2}\left(1 - \frac{\Delta m^2\cos 2\theta - a}{\sqrt{(\Delta m^2\cos 2\theta - a)^2 + (\Delta m^2\sin 2\theta)^2}}\right)$$

In General

Two flavor:

$$|\widehat{U}_{\alpha i}|^2 = \frac{\widehat{m^2}_i - \xi_\alpha}{\Delta\widehat{m^2}_{ij}}$$

leads to

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$$= \frac{1}{2}\left(1 - \frac{\Delta m^2\cos 2\theta - a}{\sqrt{(\Delta m^2\cos 2\theta - a)^2 + (\Delta m^2\sin 2\theta)^2}}\right)$$

Numerically checked for $N = 4, 5$.

True for all N?

We did it!

EIGENVECTORS FROM EIGENVALUES

PETER B. DENTON, STEPHEN J. PARKE, TERENCE TAO, AND XINING ZHANG

ABSTRACT. We present a new method of succinctly determining eigenvectors from eigenvalues. Specifically, we relate the norm squared of the elements of eigenvectors to the eigenvalues and the submatrix eigenvalues.

$$|v_{i,j}|^2 = \frac{\prod_{k=1}^{n-1}(\lambda_i - \xi_{j,k})}{\prod_{k=1;k\neq i}^{n}(\lambda_i - \lambda_k)}$$

Key Points

- Someone write a paper about the moon with Tom
- Tom gave me enough freedom to learn how to research
- Tom pushed me to go elsewhere

- All you need is ~~Love~~ Eigenvalues
- Terry is "cheery firehose"