Abstract

As long-baseline efforts are ramped up over coming years, it is important understand how the presence of matter affects neutrino oscillations. In this talk I will discuss precision oscillation probability formulas with matter effects. We have developed expressions that are simple, precise, and an actual expansion in the small parameters: $\sin^2\theta_{13}$ and $\Delta m^2_{21}/\Delta m^2_{31}$. In addition, our expressions return to the exact expression in vacuum. I will also present some recent results on understanding CP violation in matter which show that the matter effect of the Jarlskog simply factorizes into atmospheric and solar contributions.

Neutrino Oscillation Probabilities in Matter

Peter B. Denton

DPF

July 29, 2019

BROOKHAVEN NATIONAL LABORATORY

NDI

Neutrino Oscillation Parameters Status

Six parameters:

1. $\theta_{13} = (8.6 \pm 0.1)^\circ$
2. $\theta_{12} = (33.8 \pm 0.8)^\circ$
3. $\Delta m^2_{21} = (7.4 \pm 0.2) \times 10^{-5}$ eV^2
4. $\theta_{23} \sim 45^\circ$ (octant)
5. $|\Delta m^2_{31}| = (2.52 \pm 0.03) \times 10^{-3}$ eV^2 (mass ordering)
6. $\delta =$???

NuFIT, 1811.05487

PMNS order allows for easy measurement of θ_{13} and θ_{12}.

θ_{23} and $\delta_{\rm CP}$ require full three-flavor description.

Analytic Oscillation Probabilities in Matter

- ☑ Solar: $P_{ee} \simeq \sin^2\theta_\odot$

  Approx: S. Mikheev, A. Smirnov, Nuovo Cim. C9 (1986) 17-26

  Exact: S. Parke, PRL 57 (1986) 2322

- ☑ Long-baseline: All three flavors

  Exact: H. Zaglauer, K. Schwarzer, Z.Phys. C40 (1988) 273

  Approx: PBD, H. Minakata, S. Parke, 1604.08167

- ☑ ν_e disappearance (neutrino factory):
  $\Delta\widehat{m^2}_{ee} = \widehat{m^2}_3 - (\widehat{m^2}_1 + \widehat{m^2}_2 - \Delta m^2_{21} c^2_{12})$

  PBD, S. Parke, 1808.09453

- ☐ Atmospheric

The Billion Dollar Question

What is $P(\nu_\mu \to \nu_e)$?

$$P(\overset{(-)}{\nu}_\mu \to \overset{(-)}{\nu}_e) = |\mathcal{A}_{\mu e}|^2 \qquad \mathcal{A}_{\mu e} = \mathcal{A}_{31} + e^{\pm i\Delta_{32}}\mathcal{A}_{21}$$

$$\mathcal{A}_{31} = 2s_{13}c_{13}s_{23}\sin\Delta_{31}$$

$$\mathcal{A}_{21} = 2s_{12}c_{13}(c_{12}c_{23}e^{i\delta} - s_{12}s_{13}s_{23})\sin\Delta_{21}$$

$$\Delta_{ij} = \Delta m^2{}_{ij}L/4E$$

The Billion Dollar Question

What is $P(\nu_\mu \to \nu_e)$?

$$P(\overset{(-)}{\nu}_\mu \to \overset{(-)}{\nu}_e) = |\mathcal{A}_{\mu e}|^2 \qquad \mathcal{A}_{\mu e} = \mathcal{A}_{31} + e^{\pm i\Delta_{32}}\mathcal{A}_{21}$$

$$\mathcal{A}_{31} = 2s_{13}c_{13}s_{23}\sin\Delta_{31}$$

$$\mathcal{A}_{21} = 2s_{12}c_{13}(c_{12}c_{23}e^{i\delta} - s_{12}s_{13}s_{23})\sin\Delta_{21}$$

$$\Delta_{ij} = \Delta m^2{}_{ij}L/4E$$

...in matter?

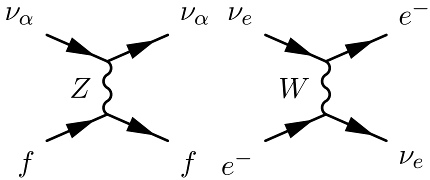
Now: NOvA, T2K, MINOS, ...
Upcoming: DUNE, T2HK, ...
Second maximum: T2HKK? ESSnuSB? ...

Matter Effects Matter

Call Schrödinger equation's eigenvalues m_i^2 and eigenvectors U_i.

$$\mathcal{A}(\nu_\alpha \to \nu_\beta) = \sum_{i=1}^{3} U_{\alpha i}^* U_{\beta i} e^{-i m_i^2 L/2E} \qquad\qquad P = |\mathcal{A}|^2$$

In matter ν's propagate in a new basis that depends on $a \propto \rho E$.

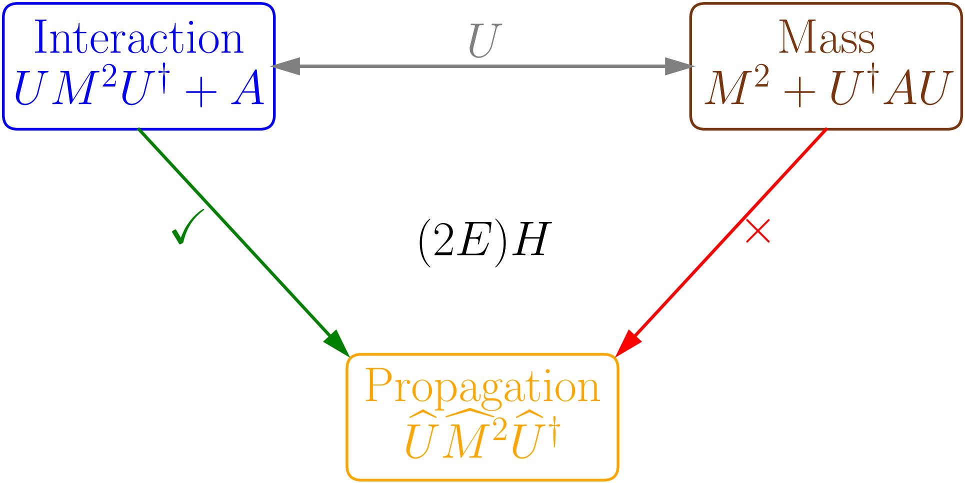


L. Wolfenstein, PRD 17 (1978)

Eigenvalues: $m_i^2 \to \widehat{m^2}_i(a)$

Eigenvectors are given by $\theta_{ij} \to \widehat{\theta}_{ij}(a) \qquad \Leftarrow \qquad$ Unitarity

Three Bases


Interaction
$UM^2U^\dagger + A$
U
Mass
$M^2 + U^\dagger AU$
✓
$(2E)H$
×
Propagation
$\widehat{U}\widehat{M^2}\widehat{U}^\dagger$

Hamiltonian Dynamics

$$H = \frac{1}{2E}\left[U\begin{pmatrix}0 & & \\ & \Delta m^2_{21} & \\ & & \Delta m^2_{31}\end{pmatrix}U^\dagger + \begin{pmatrix}a & & \\ & 0 & \\ & & 0\end{pmatrix}\right]$$

$a = 2\sqrt{2}G_F N_e E$

$$U = \begin{pmatrix}1 & & \\ & c_{23} & s_{23} \\ & -s_{23} & c_{23}\end{pmatrix}\begin{pmatrix}c_{13} & & s_{13}e^{-i\delta} \\ & 1 & \\ -s_{13}e^{i\delta} & & c_{13}\end{pmatrix}\begin{pmatrix}c_{12} & s_{12} & \\ -s_{12} & c_{12} & \\ & & 1\end{pmatrix}$$

Find eigenvalues and eigenvectors:

$$H = \frac{1}{2E}\widehat{U}\begin{pmatrix}0 & & \\ & \widehat{\Delta m^2}_{21} & \\ & & \widehat{\Delta m^2}_{31}\end{pmatrix}\widehat{U}^\dagger$$

Computationally works, but we can do better than a $\boxed{\textbf{black box}}$...

Eigenvalues Analytically: The Exact Solution

Solve the cubic characteristic equation.

G. Cardano *Ars Magna* 1545

V. Barger, et al., PRD 22 (1980) 2718

H. Zaglauer, K. Schwarzer, Z.Phys. C40 (1988) 273

Eigenvalues ($\widehat{m^2}_i$) and eigenvectors ($\widehat{\theta}$, $\widehat{\delta}$) depend on S:

$$S = \cos\left\{\frac{1}{3}\cos^{-1}\left[\frac{2A^3 - 9AB + 27C}{2\left(A^2 - 3B\right)^{3/2}}\right]\right\}$$

A, B, and C depend on vacuum parameters and matter potential

Traded one $\boxed{\textbf{black box}}$ for another...

We're physicists so ...

Perturbation theory

Alternative Solutions

- Small matter potential: $a/\Delta m^2$

  Y. Li, Y. Wang, Z-z. Xing, 1605.00900

- s_{13}, s_{13}^2

  A. Cervera, et al., hep-ph/0002108

  H. Minakata, 0910.5545

  K. Asano, H. Minakata, 1103.4387

- $\Delta m_{21}^2/\Delta m_{31}^2 \sim 0.03$

  J. Arafune, J. Sato, hep-ph/9607437

  A. Cervera, et al., hep-ph/0002108

  M. Freund, hep-ph/0103300

  E. Akhmedov, et al., hep-ph/0402175

  S. Agarwalla, Y. Kao, T. Takeuchi, 1302.6773

  M. Blennow, A. Smirnov, 1306.2903

  H. Minakata, S. Parke, 1505.01826

  PBD, H. Minakata, S. Parke, 1604.08167

(See G. Barenboim, PBD, S. Parke, C. Ternes 1902.00517 for a review)

A Tale of Two Tools

Split the Hamiltonian into:

- Large, diagonal part (H_0)
- Small, *off-diagonal* part (H_1)

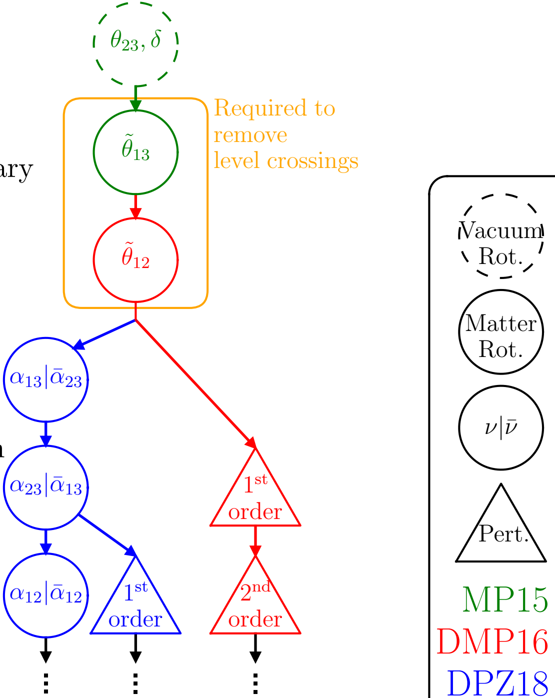
- Improves precision at zeroth order
- Naturally leads to using $\Delta m^2_{ee} \equiv c^2_{12}\Delta m^2_{31} + s^2_{12}\Delta m^2_{32}$

H. Nunokawa, S. Parke, R. Zukanovich, hep-ph/0503283

1. **Rotations**:
   - A two-flavor rotation only requires solving a quadratic
   - Diagonalize away the big off-diagonal terms
   - Follows the order of the PMNS matrix
2. **Perturbative expansion**:
   - Smallness parameter is $|\epsilon'| \leq 0.015$
   - Correct eigenvalues ($\widetilde{m^2}_i$) and eigenvectors ($\widetilde{\theta_{ij}}$)
   - Eigenvalues already include 1$^{\text{st}}$ order corrections at 0$^{\text{th}}$ order
   - Can improve the precision to arbitrary order

Roadmap

- Two rotations are necessary
- Order is lucky



- Further precision through perturbation theory or rotations

Probability in Matter: DMP 0^{th}

Matter expression $\Rightarrow$ Vacuum expression

$$\widetilde{P}_{\alpha\beta}(\Delta m^2_{21}, \Delta m^2_{31}, \theta_{13}, \theta_{12}, \theta_{23}, \delta) = P_{\alpha\beta}(\Delta \widetilde{m^2}_{21}, \Delta \widetilde{m^2}_{31}, \widetilde{\theta}_{13}, \widetilde{\theta}_{12}, \theta_{23}, \delta)$$

Same expression, 4 new variables.

Probability in Matter: DMP 0^{th}

Matter expression $\Rightarrow$ Vacuum expression

$$\widetilde{P}_{\alpha\beta}(\Delta m^2_{21}, \Delta m^2_{31}, \theta_{13}, \theta_{12}, \theta_{23}, \delta) = P_{\alpha\beta}(\Delta\widetilde{m^2}_{21}, \Delta\widetilde{m^2}_{31}, \widetilde{\theta}_{13}, \widetilde{\theta}_{12}, \theta_{23}, \delta)$$

Same expression, 4 new variables.

$$\cos 2\widetilde{\theta}_{13} = \frac{\Delta m^2_{ee}\cos 2\theta_{13} - a}{\Delta\widetilde{m^2}_{ee}}$$

$$\Delta\widetilde{m^2}_{ee} = \Delta m^2_{ee}\sqrt{(\cos 2\theta_{13} - a/\Delta m^2_{ee})^2 + \sin^2 2\theta_{13}}$$

See also PBD, S. Parke, 1808.09453

Probability in Matter: DMP 0^{th}

Matter expression $\Rightarrow$ Vacuum expression

$$\widetilde{P}_{\alpha\beta}(\Delta m^2_{21},\Delta m^2_{31},\theta_{13},\theta_{12},\theta_{23},\delta)=P_{\alpha\beta}(\Delta\widetilde{m^2}_{21},\Delta\widetilde{m^2}_{31},\widetilde{\theta}_{13},\widetilde{\theta}_{12},\theta_{23},\delta)$$

Same expression, 4 new variables.

$$\cos 2\widetilde{\theta}_{13}=\frac{\Delta m^2_{ee}\cos 2\theta_{13}-a}{\Delta\widetilde{m^2}_{ee}}$$

$$\Delta\widetilde{m^2}_{ee}=\Delta m^2_{ee}\sqrt{(\cos 2\theta_{13}-a/\Delta m^2_{ee})^2+\sin^2 2\theta_{13}}$$

See also PBD, S. Parke, 1808.09453

$$\cos 2\widetilde{\theta}_{12}=\frac{\Delta m^2_{21}\cos 2\theta_{12}-a_{12}}{\Delta\widetilde{m^2}_{21}}\,,\qquad a_{12}=(a+\Delta m^2_{ee}-\Delta\widetilde{m^2}_{ee})/2$$

$$\Delta\widetilde{m^2}_{21}=\Delta m^2_{21}\sqrt{(\cos 2\theta_{12}-a_{12}/\Delta m^2_{21})^2+\cos^2(\widetilde{\theta}_{13}-\theta_{13})\sin^2 2\theta_{12}}$$

Probability in Matter: DMP 0^{th}

Matter expression $\Rightarrow$ Vacuum expression

$$\widetilde{P}_{\alpha\beta}(\Delta m^2_{21}, \Delta m^2_{31}, \theta_{13}, \theta_{12}, \theta_{23}, \delta) = P_{\alpha\beta}(\widetilde{\Delta m^2}_{21}, \widetilde{\Delta m^2}_{31}, \widetilde{\theta}_{13}, \widetilde{\theta}_{12}, \theta_{23}, \delta)$$

Same expression, 4 new variables.

$$\cos 2\widetilde{\theta}_{13} = \frac{\Delta m^2_{ee}\cos 2\theta_{13} - a}{\widetilde{\Delta m^2}_{ee}}$$

$$\widetilde{\Delta m^2}_{ee} = \Delta m^2_{ee}\sqrt{(\cos 2\theta_{13} - a/\Delta m^2_{ee})^2 + \sin^2 2\theta_{13}}$$

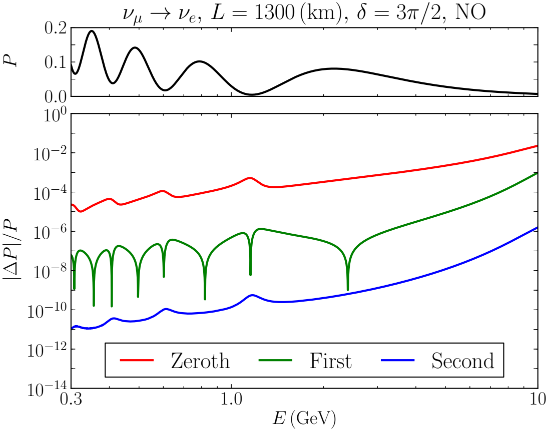
See also PBD, S. Parke, 1808.09453

$$\cos 2\widetilde{\theta}_{12} = \frac{\Delta m^2_{21}\cos 2\theta_{12} - a_{12}}{\widetilde{\Delta m^2}_{21}}, \qquad a_{12} = (a + \Delta m^2_{ee} - \widetilde{\Delta m^2}_{ee})/2$$

$$\widetilde{\Delta m^2}_{21} = \Delta m^2_{21}\sqrt{(\cos 2\theta_{12} - a_{12}/\Delta m^2_{21})^2 + \cos^2(\widetilde{\theta}_{13} - \theta_{13})\sin^2 2\theta_{12}}$$

$$\widetilde{\Delta m^2}_{31} = \Delta m^2_{31} + \frac{1}{4}a + \frac{1}{2}(\widetilde{\Delta m^2}_{21} - \Delta m^2_{21}) + \frac{3}{4}(\widetilde{\Delta m^2}_{ee} - \Delta m^2_{ee})$$

Precision



DUNE: NO, $\delta = 3\pi/2$		First min	First max
$P(\nu_\mu \to \nu_e)$		0.0047	0.081
E (GeV)		1.2	2.2
$\frac{\lvert\Delta P\rvert}{P}$	Zeroth	5×10^{-4}	4×10^{-4}
	First	3×10^{-7}	2×10^{-7}
	Second	6×10^{-10}	5×10^{-10}

The CPV Term in Matter

The amount of CPV is

$$P_{\alpha\beta} - \bar{P}_{\alpha\beta} = \pm 16 J \sin\Delta_{21} \sin\Delta_{31} \sin\Delta_{32} \qquad \alpha \neq \beta$$

where the Jarlskog is

$$J \equiv \Im[U_{e1} U_{\mu 2} U_{e2}^* U_{\mu 1}^*]$$
$$J = c_{12} s_{12} c_{13}^2 s_{13} c_{23} s_{23} \sin\delta$$

C. Jarlskog, PRL 55 (1985)

The exact term in matter is known to be

$$\frac{\widehat{J}}{J} = \frac{\Delta m_{21}^2 \Delta m_{31}^2 \Delta m_{32}^2}{\Delta\widehat{m^2}_{21} \Delta\widehat{m^2}_{31} \Delta\widehat{m^2}_{32}}$$

V. Naumov, IJMP 1992

P. Harrison, W. Scott, hep-ph/9912435

Our approximation reproduces this order by order in ϵ'

CPV in Matter

CPV in matter can be written sans $\cos(\frac{1}{3}\cos^{-1}(\cdots))$ term.

$$\frac{\widehat{J}}{J}=\frac{\Delta m^2_{21}\Delta m^2_{31}\Delta m^2_{32}}{\Delta\widehat{m^2}_{21}\Delta\widehat{m^2}_{31}\Delta\widehat{m^2}_{32}}$$

$$\left(\Delta\widehat{m^2}_{21}\Delta\widehat{m^2}_{31}\Delta\widehat{m^2}_{32}\right)^2=(A^2-4B)(B^2-4AC)+(2AB-27C)C$$

$$A\equiv\sum_j\widehat{m^2}_j=\Delta m^2_{31}+\Delta m^2_{21}+a$$

$$B\equiv\sum_{j>k}\widehat{m^2}_j\widehat{m^2}_k=\Delta m^2_{31}\Delta m^2_{21}+a(\Delta m^2_{ee}c^2_{13}+\Delta m^2_{21})$$

$$C\equiv\prod_j\widehat{m^2}_j=a\Delta m^2_{31}\Delta m^2_{21}c^2_{13}c^2_{12}$$

This is the *only* oscillation quantity in matter that can be written exactly without $\cos(\frac{1}{3}\cos^{-1}(\cdots))$!

CPV in Matter

Thus $\widehat{J}^2$ is fourth order in matter potential:
only two matter corrections are needed.

$$\frac{\widehat{J}}{J} = \frac{1}{|1-(a/\alpha_1)e^{i2\theta_1}||1-(a/\alpha_2)e^{i2\theta_2}|}$$

CPV in Matter

Thus $\widehat{J}^2$ is fourth order in matter potential:
only two matter corrections are needed.

$$\frac{\widehat{J}}{J}=\frac{1}{|1-(a/\alpha_1)e^{i2\theta_1}||1-(a/\alpha_2)e^{i2\theta_2}|}$$

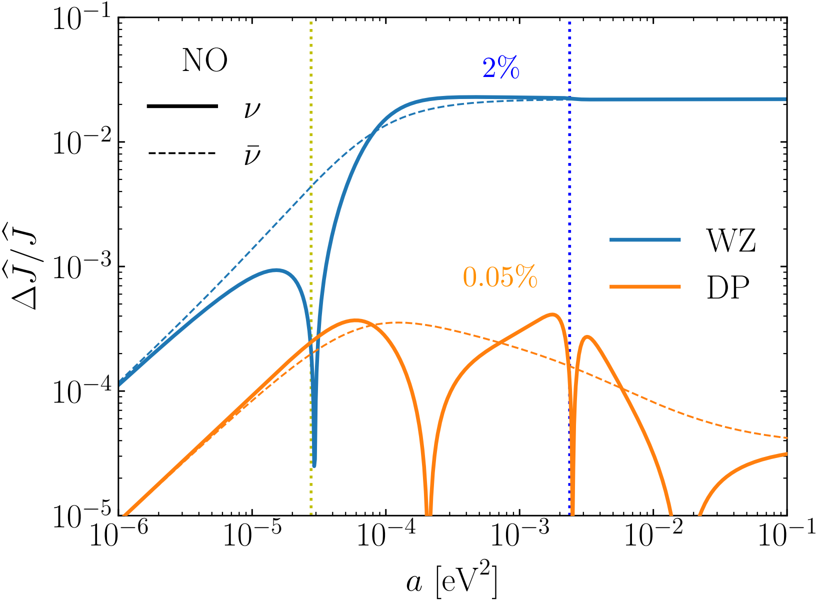
CPV in matter can be well approximated:

$$\frac{\widehat{J}}{J}\approx\frac{1}{|1-(a/\Delta m^2_{ee})e^{i2\theta_{13}}||1-(c^2_{13}a/\Delta m^2_{21})e^{i2\theta_{12}}|}$$

PBD, Parke, 1902.07185

See also X. Wang, S. Zhou, 1901.10882

CPV In Matter Approximation Precision



New Physics

DUNE and T2HK will unprecedented capabilities to test the three-neutrino oscillation picture

Extend DMP to new physics progress report:

- ☑ Sterile

S. Parke, X. Zhang, 1905.01356

See Xining's talk tomorrow at 4:45 here!

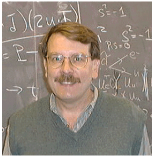
- ☐ NSI
- ☐ Neutrino decay
- ☐ Decoherence
- ☐ …

Key Points

- Long-baseline oscillations are fundamentally three-flavor
- Rotate **large terms first** $\Rightarrow$ PMNS order, removes level crossings
- 0^{th} order probabilities: **same structure as vacuum** probabilities
- 0^{th} order: **accurate** enough for current & future experiments
- Exact and approximate CPV in matter are **simpler** than expected

Backups

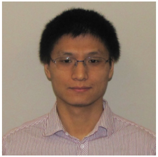
Analytic Oscillation Probability Collaborators



Stephen Parke

Hisakazu Minakata

Gabriela Barenboim



Xining Zhang

Christoph Ternes

1604.08167, 1806.01277, 1808.09453,
1902.00517, 1902.07185, 1907.02534
github.com/PeterDenton/Nu-Pert
github.com/PeterDenton/Nu-Pert-Compare

Variable Matter Density

We assume ρ is constant. Is this okay?

If ρ varies only "slowly," we can set ρ to the average:

$$\rho(x) \to \bar{\rho} = \frac{1}{L}\int_0^L \rho(x)dx$$

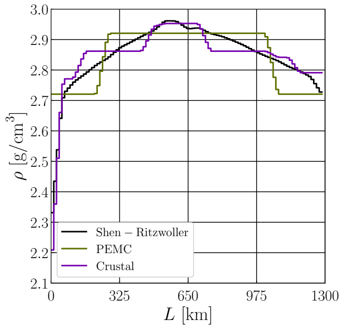
ρ doesn't vary "too much" when

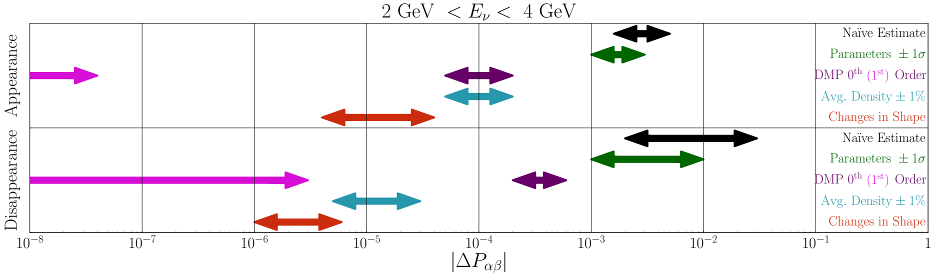
$$\left|\frac{d\widehat{\theta}}{dt}\right| \ll \left|\frac{\Delta\widehat{m^2}}{2E}\right|$$

True for DUNE?

Variable Matter Density

This is a great approximation at DUNE: ✓!





K. Kelly, S. Parke, 1802.06784

Eigenvalues Analytically: The Exact Solution

Solve the cubic characteristic equation.

$$\widehat{m^2}_1 = \frac{A}{3} - \frac{1}{3}\sqrt{A^2 - 3B}\,S - \frac{\sqrt{3}}{3}\sqrt{A^2 - 3B}\sqrt{1 - S^2}$$

$$\widehat{m^2}_2 = \frac{A}{3} - \frac{1}{3}\sqrt{A^2 - 3B}\,S + \frac{\sqrt{3}}{3}\sqrt{A^2 - 3B}\sqrt{1 - S^2}$$

$$\widehat{m^2}_3 = \frac{A}{3} + \frac{2}{3}\sqrt{A^2 - 3B}\,S$$

$$A = \Delta m^2_{21} + \Delta m^2_{31} + a$$

$$B = \Delta m^2_{21}\Delta m^2_{31} + a\left[c^2_{13}\Delta m^2_{31} + (c^2_{12}c^2_{13} + s^2_{13})\Delta m^2_{21}\right]$$

$$C = a\Delta m^2_{21}\Delta m^2_{31}c^2_{12}c^2_{13}$$

$$S = \cos\left\{\frac{1}{3}\cos^{-1}\left[\frac{2A^3 - 9AB + 27C}{2\left(A^2 - 3B\right)^{3/2}}\right]\right\}$$

H. Zaglauer, K. Schwarzer, Z.Phys. C40 (1988) 273

Traded one **black box** for another...

Alternative Solutions: Example

$$P_0 = \sin^2\theta_{23} \frac{\sin^2 2\theta_{13}}{\hat{C}^2} \sin^2(\hat{\Delta}\hat{C}), \tag{36a}$$

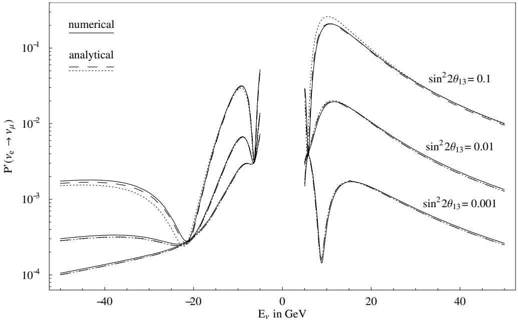
$$P_{\sin\delta} = \frac{1}{2}\alpha \frac{\sin\delta \cos\theta_{13} \sin 2\theta_{12} \sin 2\theta_{13} \sin 2\theta_{23}}{\hat{A}\hat{C}\cos\theta_{13}^2} \sin(\hat{C}\hat{\Delta}) \times \{\cos(\hat{C}\hat{\Delta}) - \cos((1+\hat{A})\hat{\Delta})\}, \tag{36b}$$

$$P_{\cos\delta} = \frac{1}{2}\alpha \frac{\cos\delta \cos\theta_{13} \sin 2\theta_{12} \sin 2\theta_{13} \sin 2\theta_{23}}{\hat{A}\hat{C}\cos\theta_{13}^2} \sin(\hat{C}\hat{\Delta}) \times \{\sin((1+\hat{A})\hat{\Delta}) \mp \sin(\hat{C}\hat{\Delta})\}, \tag{36c}$$

$$P_1 = -\alpha \frac{1-\hat{A}\cos 2\theta_{13}}{\hat{C}^3} \sin^2\theta_{12} \sin^2 2\theta_{13} \sin^2\theta_{23}\hat{\Delta} \times \sin(2\hat{\Delta}\hat{C}) + \alpha \frac{2\hat{A}(-\hat{A}+\cos 2\theta_{13})}{\hat{C}^4} \times \sin^2\theta_{12} \sin^2 2\theta_{13} \sin^2\theta_{23} \sin^2(\hat{\Delta}\hat{C}), \tag{36d}$$

$$P_2 = \alpha \frac{\mp 1 + \hat{C} \pm \hat{A}\cos 2\theta_{13}}{2\hat{C}^2\hat{A}\cos^2\theta_{13}} \cos\theta_{13} \sin 2\theta_{12} \sin 2\theta_{13} \times \sin 2\theta_{23} \sin^2(\hat{\Delta}\hat{C}), \tag{36e}$$

$$P_3 = \alpha^2 \frac{2\hat{C}\cos^2\theta_{23}\sin^2 2\theta_{12}}{\hat{A}^2\cos^2\theta_{13}(\mp\hat{A}+\hat{C}\pm\cos 2\theta_{13})} \times \sin^2\left(\frac{1}{2}(1+\hat{A}\mp\hat{C})\hat{\Delta}\right). \tag{36f}$$



M. Freund, hep-ph/0103300

“What is Δm^2_{ee}?”

$$\Delta m^2_{ee} = c^2_{12}\Delta m^2_{31} + s^2_{12}\Delta m^2_{32} = \Delta m^2_{31} - s^2_{12}\Delta m^2_{21}$$

H. Nunokawa, S. Parke, R. Funchal, hep-ph/0503283

S. Parke, 1601.07464

Additional expressions for $\Delta m^2_{\mu\mu}, \Delta m^2_{\tau\tau}$

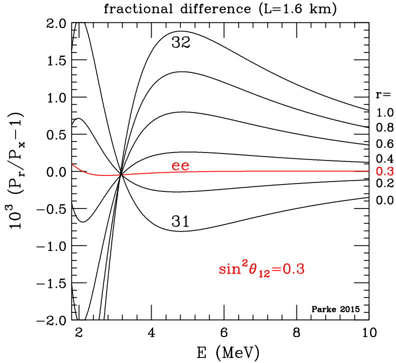
Useful definitions:

- ν_e weighted average of atmospheric splittings:

$$m_3^2 - \frac{|U_{e1}|^2 m_1^2 + |U_{e2}|^2 m_2^2}{|U_{e1}|^2 + |U_{e2}|^2}$$

- Measured by reactor experiments with smallest L/E error
- Simple form:
$P_{ee} \approx 1 - \sin^2 2\theta_{13} \sin^2 \Delta_{ee}$

$$\Delta_{ij} = \Delta m^2{}_{ij} L/4E$$



Neutrino Oscillations in Vacuum: Disappearance

It is easy to calculate the *exact* disappearance expression in vacuum:

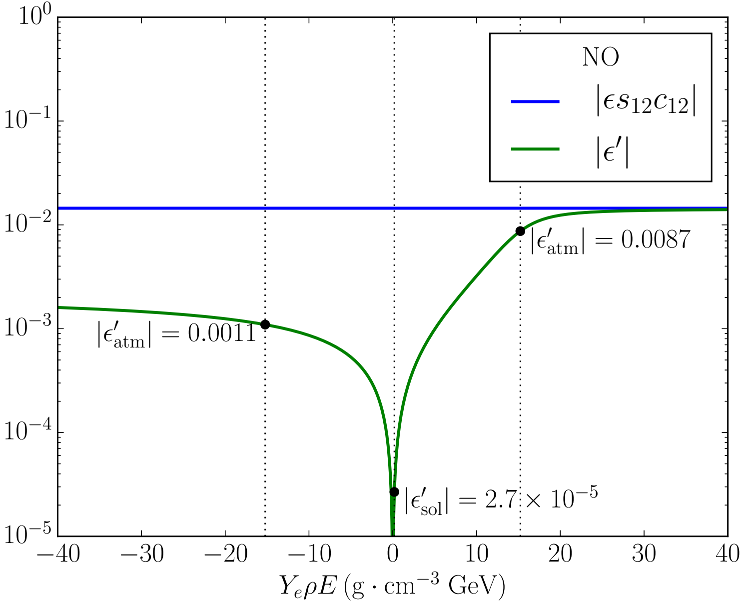
$$P(\nu_\alpha \to \nu_\alpha) = 1 - 4 \sum_{i<j} |U_{\alpha i}|^2 |U_{\alpha j}|^2 \sin^2 \Delta_{ji}$$

For the electron case this expression is simple:

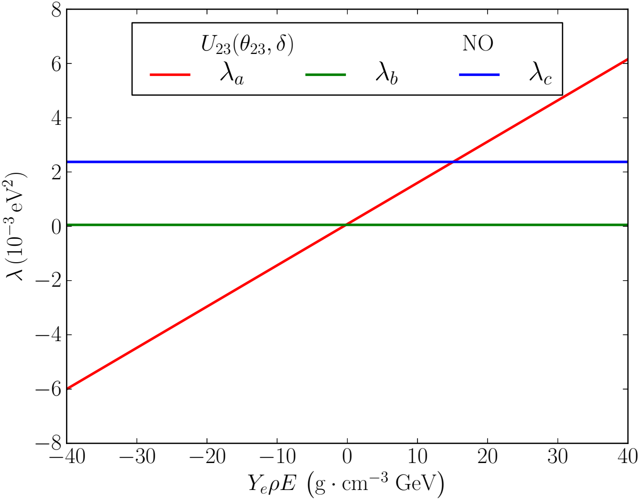
$$\begin{aligned} P(\nu_e \to \nu_e) = 1 & \\ & - 4c_{12}^2 s_{12}^2 c_{13}^4 \sin^2 \Delta_{21} \\ & - 4c_{12}^2 c_{13}^2 s_{13}^2 \sin^2 \Delta_{31} \\ & - 4s_{12}^2 c_{13}^2 s_{13}^2 \sin^2 \Delta_{32} \end{aligned}$$

$$\Delta_{ij} = \frac{\Delta m_{ij}^2 L}{4E}$$
$$\Delta m_{ij}^2 = m_i^2 - m_j^2$$

Expansion Parameter

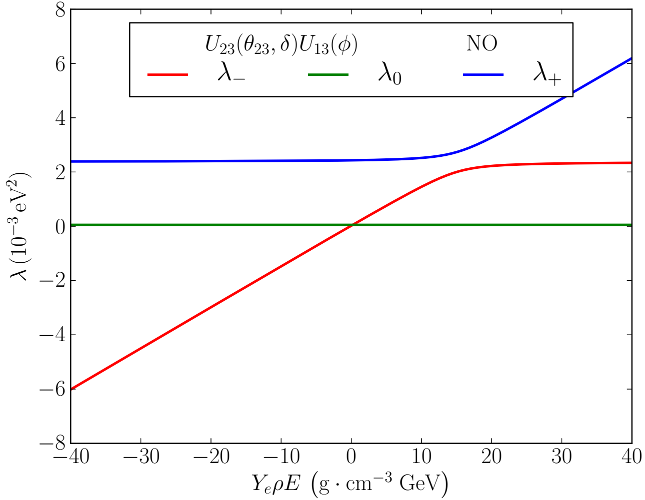


Eigenvalues in Matter: Two Rotations are Needed



$$\widetilde{m^2}_a = a + (s_{13}^2 + \epsilon s_{12}^2)\Delta m_{ee}^2\,,\ \widetilde{m^2}_b = \epsilon c_{12}^2 \Delta m_{ee}^2\,,\ \widetilde{m^2}_c = (c_{13}^2 + \epsilon s_{12}^2)\Delta m_{ee}^2$$

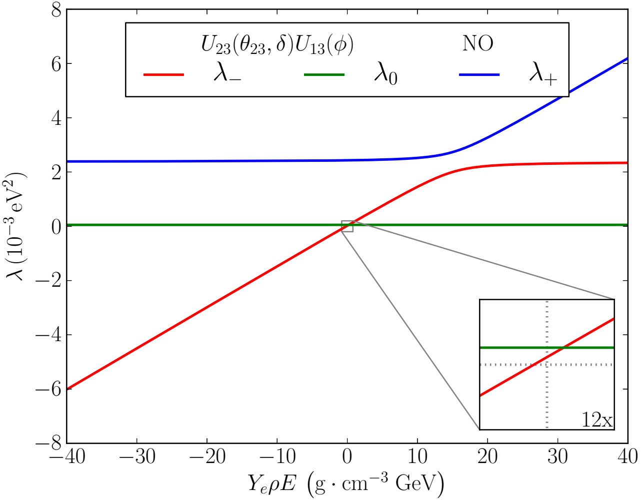
Eigenvalues in Matter: Two Rotations are Needed



$$\widetilde{m^2}_{\mp} = \frac{1}{2}\left[(\widetilde{m^2}_a + \widetilde{m^2}_c) \mp \mathrm{sgn}(\Delta m^2_{ee})\sqrt{(\widetilde{m^2}_c - \widetilde{m^2}_a)^2 + (2s_{13}c_{13}\Delta m^2_{ee})^2}\right]$$

$$\widetilde{m^2}_0 = \widetilde{m^2}_b$$

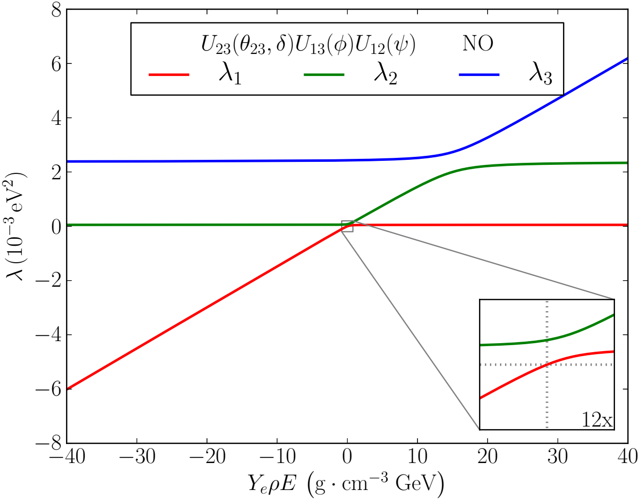
Eigenvalues in Matter: Two Rotations are Needed



$$\widetilde{m^2}_{\mp} = \frac{1}{2}\left[(\widetilde{m^2}_a + \widetilde{m^2}_c) \mp \mathrm{sgn}(\Delta m^2_{ee})\sqrt{(\widetilde{m^2}_c - \widetilde{m^2}_a)^2 + (2 s_{13} c_{13} \Delta m^2_{ee})^2}\right]$$

$$\widetilde{m^2}_0 = \widetilde{m^2}_b$$

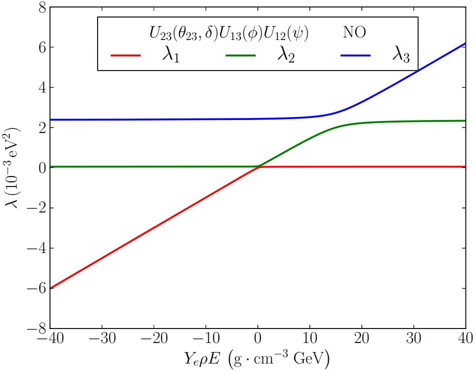
Eigenvalues in Matter: Two Rotations are Needed

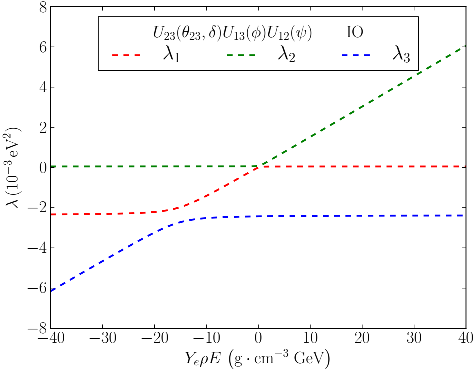


$$\widetilde{m^2}_{1,2} = \frac{1}{2}\left[(\widetilde{m^2}_0 + \widetilde{m^2}_-) \mp \sqrt{(\widetilde{m^2}_0 - \widetilde{m^2}_-)^2 + (2\epsilon c_{(\widetilde{\theta}_{13}-\theta_{13})} c_{12} s_{12} \Delta m^2_{ee})^2}\right]$$

$$\widetilde{m^2}_3 = \widetilde{m^2}_+$$

Eigenvalues in Matter: Mass Ordering





NO

$$\widetilde{m^2}_1 < \widetilde{m^2}_2 < \widetilde{m^2}_3$$

IO

$$\widetilde{m^2}_3 < \widetilde{m^2}_1 < \widetilde{m^2}_2$$

Exact Neutrino Oscillations in Matter: Mixing Angles

$$s^2_{\widehat{12}} = \frac{-\left[(\widehat{m^2}_2)^2 - \alpha\widehat{m^2}_2 + \beta\right]\Delta\widetilde{m^2}_{31}}{\left[(\widehat{m^2}_1)^2 - \alpha\widehat{m^2}_1 + \beta\right]\Delta\widetilde{m^2}_{32} - \left[(\widehat{m^2}_2)^2 - \alpha\widehat{m^2}_2 + \beta\right]\Delta\widetilde{m^2}_{31}}$$

$$s^2_{\widehat{13}} = \frac{(\widehat{m^2}_3)^2 - \alpha\widehat{m^2}_3 + \beta}{\Delta\widetilde{m^2}_{31}\Delta\widetilde{m^2}_{32}}$$

$$s^2_{\widehat{23}} = \frac{s^2_{23}E^2 + c^2_{23}F^2 + 2c_{23}s_{23}c_\delta EF}{E^2 + F^2}$$

$$e^{-i\widehat{\delta}} = \frac{c^2_{23}s^2_{23}\left(e^{-i\delta}E^2 - e^{i\delta}F^2\right) + \left(c^2_{23} - s^2_{23}\right)EF}{\sqrt{\left(s^2_{23}E^2 + c^2_{23}F^2 + 2EFc_{23}s_{23}c_\delta\right)\left(c^2_{23}E^2 + s^2_{23}F^2 - 2EFc_{23}s_{23}c_\delta\right)}}$$

$$\alpha = c^2_{13}\Delta m^2_{31} + \left(c^2_{12}c^2_{13} + s^2_{13}\right)\Delta m^2_{21}\,, \;\; \beta = c^2_{12}c^2_{13}\Delta m^2_{21}\Delta m^2_{31}$$

$$E = c_{13}s_{13}\left[\left(\widehat{m^2}_3 - \Delta m^2_{21}\right)\Delta m^2_{31} - s^2_{12}\left(\widehat{m^2}_3 - \Delta m^2_{31}\right)\Delta m^2_{21}\right]$$

$$F = c_{12}s_{12}c_{13}\left(\widehat{m^2}_3 - \Delta m^2_{31}\right)\Delta m^2_{21}$$

H. Zaglauer, K. Schwarzer, Z.Phys. C40 (1988) 273

$\widetilde{m^2}_{1,2} - \widetilde{\theta}_{12}$ Symmetry

From the shape of $U_{12}(\widetilde{\theta}_{12})$, it is clear that the probabilities are invariant under a simultaneous interchange of

$$\widetilde{m^2}_1 \leftrightarrow \widetilde{m^2}_2\,, \qquad \text{and} \qquad \widetilde{\theta}_{12} \to \widetilde{\theta}_{12} \pm \frac{\pi}{2}\,.$$

Since only even powers of $\widetilde{\theta}_{12}$ trig functions $c^2_{\widetilde{12}}, s^2_{\widetilde{12}}, c_{\widetilde{12}}s_{\widetilde{12}}, \cos(2\widetilde{\theta}_{12}), \sin(2\widetilde{\theta}_{12})$ appear in the probabilities, the sign degeneracy is irrelevant.

More usefully, we can write that the probabilities are invariant under the simultaneous interchange of

$$\widetilde{m^2}_1 \leftrightarrow \widetilde{m^2}_2\,, \qquad c^2_{\widetilde{12}} \leftrightarrow s^2_{\widetilde{12}}\,, \qquad \text{and} \qquad c_{\widetilde{12}}s_{\widetilde{12}} \to -c_{\widetilde{12}}s_{\widetilde{12}}\,.$$

This interchange constrains the $\sin^2\Delta_{21}$ term,
and the $\sin^2\Delta_{32}$ term easily follows from the $\sin^2\Delta_{31}$ term.

General Form of the First Order Coefficients

$$P_{\alpha\beta} = \delta_{\alpha\beta} + 4C_{21}^{\alpha\beta}\sin^2\Delta_{21} + 4C_{31}^{\alpha\beta}\sin^2\Delta_{31} + 4C_{32}^{\alpha\beta}\sin^2\Delta_{32} + 8D^{\alpha\beta}\sin\Delta_{21}\sin\Delta_{31}\sin\Delta_{32}$$

Can reduce 8 expressions down to 3:

$$(C_{21}^{\alpha\beta})^{(1)} = \epsilon'\Delta m_{ee}^2\left(\frac{F_1^{\alpha\beta}}{\widetilde{\Delta m^2}_{31}} + \frac{F_2^{\alpha\beta}}{\widetilde{\Delta m^2}_{32}}\right)$$

$$(C_{31}^{\alpha\beta})^{(1)} = \epsilon'\Delta m_{ee}^2\left(\frac{F_1^{\alpha\beta}+G_1^{\alpha\beta}}{\widetilde{\Delta m^2}_{31}} - \frac{F_2^{\alpha\beta}}{\widetilde{\Delta m^2}_{32}}\right)$$

$$(C_{32}^{\alpha\beta})^{(1)} = \epsilon'\Delta m_{ee}^2\left(-\frac{F_1^{\alpha\beta}}{\widetilde{\Delta m^2}_{31}} + \frac{F_2^{\alpha\beta}+G_2^{\alpha\beta}}{\widetilde{\Delta m^2}_{32}}\right)$$

$$(D^{\alpha\beta})^{(1)} = \epsilon'\Delta m_{ee}^2\left(\frac{K_1^{\alpha\beta}}{\widetilde{\Delta m^2}_{31}} - \frac{K_2^{\alpha\beta}}{\widetilde{\Delta m^2}_{32}}\right)$$

$$K_1^{\alpha\beta} = \mp s_{23}c_{23}c_{\widetilde{13}}s_{\widetilde{12}}^2(c_{\widetilde{13}}^2c_{\widetilde{12}}^2 - s_{\widetilde{13}}^2)s_\delta\,,\quad \alpha\neq\beta$$

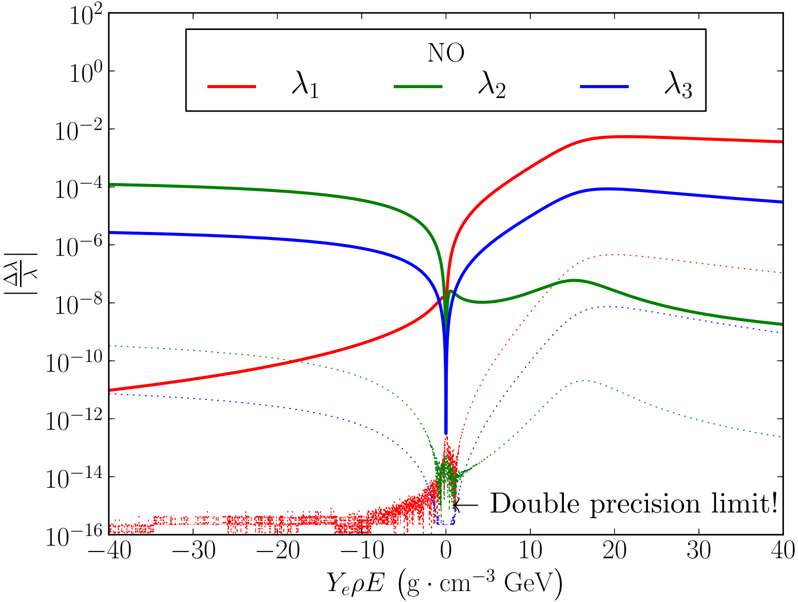
First Order Coefficients

$\nu_\alpha \to \nu_\beta$	$F_1^{\alpha\beta}$
$\nu_e \to \nu_e$	$-2c_{\widetilde{13}}^3 s_{\widetilde{13}} s_{\widetilde{12}}^3 c_{\widetilde{12}}$
$\nu_\mu \to \nu_e$	$c_{\widetilde{13}} s_{\widetilde{12}}^2 [s_{\widetilde{13}} s_{\widetilde{12}} c_{\widetilde{12}} (c_{23}^2 + c_{2\widetilde{13}} s_{23}^2)$ $-s_{23} c_{23} (s_{\widetilde{13}}^2 s_{\widetilde{12}}^2 + c_{2\widetilde{13}} c_{\widetilde{12}}^2) c_\delta]$
$\nu_\mu \to \nu_\mu$	$2c_{\widetilde{13}} s_{\widetilde{12}} (s_{23}^2 s_{\widetilde{13}} c_{\widetilde{12}} + s_{23} c_{23} s_{\widetilde{12}} c_\delta) \times$ $(c_{23}^2 c_{\widetilde{12}}^2 - 2 s_{23} c_{23} s_{\widetilde{13}} s_{\widetilde{12}} c_{\widetilde{12}} c_\delta + s_{23}^2 s_{\widetilde{13}}^2 s_{\widetilde{12}}^2)$

$\nu_\alpha \to \nu_\beta$	$G_1^{\alpha\beta}$
$\nu_e \to \nu_e$	$2 s_{\widetilde{13}} c_{\widetilde{13}} s_{\widetilde{12}} c_{\widetilde{12}} c_{2\widetilde{13}}$
$\nu_\mu \to \nu_e$	$-2 s_{\widetilde{13}} c_{\widetilde{13}} s_{\widetilde{12}} (s_{23}^2 c_{2\widetilde{13}} c_{\widetilde{12}} - s_{23} c_{23} s_{\widetilde{13}} s_{\widetilde{12}} c_\delta)$
$\nu_\mu \to \nu_\mu$	$-2 c_{\widetilde{13}} s_{\widetilde{12}} (s_{23}^2 s_{\widetilde{13}} c_{\widetilde{12}} + s_{23} c_{23} s_{\widetilde{12}} c_\delta)$ $\times (1 - 2 c_{\widetilde{13}}^2 s_{23}^2)$

Three channels gives them all with unitarity!

Eigenvalues: Precision



Hamiltonians

After a constant (θ_{23},δ) rotation, $2E\tilde{H} =$

$$\begin{pmatrix} \widetilde{m^2}_a & & s_{13}c_{13}\Delta m^2_{ee} \\ & \widetilde{m^2}_b & \\ s_{13}c_{13}\Delta m^2_{ee} & & \widetilde{m^2}_c \end{pmatrix} + \epsilon s_{12}c_{12}\Delta m^2_{ee} \begin{pmatrix} & c_{13} & \\ c_{13} & & -s_{13} \\ & -s_{13} & \end{pmatrix}$$

After a $U_{13}(\widetilde{\theta}_{13})$ rotation, $2E\hat{H} =$

$$\begin{pmatrix} \widetilde{m^2}_- & & \\ & \widetilde{m^2}_0 & \\ & & \widetilde{m^2}_+ \end{pmatrix} + \epsilon c_{12}s_{12}\Delta m^2_{ee} \begin{pmatrix} & c_{(\widetilde{\theta}_{13}-\theta_{13})} & \\ c_{(\widetilde{\theta}_{13}-\theta_{13})} & & s_{(\widetilde{\theta}_{13}-\theta_{13})} \\ & s_{(\widetilde{\theta}_{13}-\theta_{13})} & \end{pmatrix}$$

After a $U_{12}(\widetilde{\theta}_{12})$ rotation, $2E\check{H} =$

$$\begin{pmatrix} \widetilde{m^2}_1 & & \\ & \widetilde{m^2}_2 & \\ & & \widetilde{m^2}_3 \end{pmatrix} + \epsilon s_{(\widetilde{\theta}_{13}-\theta_{13})} s_{12}c_{12}\Delta m^2_{ee} \begin{pmatrix} & & -s_{\widetilde{12}} \\ & & c_{\widetilde{12}} \\ -s_{\widetilde{12}} & c_{\widetilde{12}} & \end{pmatrix}$$

Perturbative Expansion

Hamiltonian: $\check{H} = \check{H}_0 + \check{H}_1$

$$\check{H}_0 = \frac{1}{2E}\begin{pmatrix} \widetilde{m^2}_1 & & \\ & \widetilde{m^2}_2 & \\ & & \widetilde{m^2}_3 \end{pmatrix}, \quad \check{H}_1 = \epsilon' \frac{\Delta m^2_{ee}}{2E}\begin{pmatrix} & & -s_{\widetilde{12}} \\ & & c_{\widetilde{12}} \\ -s_{\widetilde{12}} & c_{\widetilde{12}} & \end{pmatrix}$$

Eigenvalues: $\widetilde{m^2}_i^{\text{ex}} = \widetilde{m^2}_i + \widetilde{m^2}_i^{(1)} + \widetilde{m^2}_i^{(2)} + \dots$

$$\widetilde{m^2}_i^{(1)} = 2E(\check{H}_1)_{ii} = 0$$

$$\widetilde{m^2}_i^{(2)} = \sum_{k\neq i} \frac{[2E(\check{H}_1)_{ik}]^2}{\Delta\widetilde{m^2}_{ik}}$$

Perturbative Expansion: Eigenvectors

Use vacuum expressions with $U \to V$ where

$$V = \widetilde{U} W$$

$\widetilde{U}$ is U with $\theta_{13} \to \widetilde{\theta}_{13}$ and $\theta_{12} \to \widetilde{\theta}_{12}$,

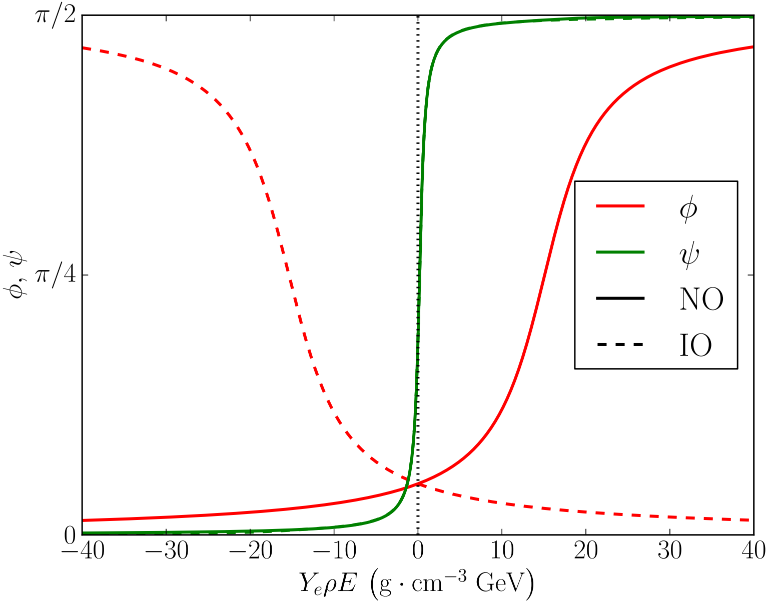
$W = W_0 + W_1 + W_2 + \dots$

$$W_0 = \mathbb{1}$$

$$W_1 = \epsilon' \Delta m^2_{ee} \begin{pmatrix} & & -\frac{s_{\widetilde{12}}}{\widetilde{\Delta m^2}_{31}} \\ & & \frac{c_{\widetilde{12}}}{\widetilde{\Delta m^2}_{32}} \\ \frac{s_{\widetilde{12}}}{\widetilde{\Delta m^2}_{31}} & -\frac{c_{\widetilde{12}}}{\widetilde{\Delta m^2}_{32}} & \end{pmatrix}$$

$$W_2 = -\epsilon'^2 \frac{(\Delta m^2_{ee})^2}{2} \begin{pmatrix} \frac{s^2_{\widetilde{12}}}{(\widetilde{\Delta m^2}_{31})^2} & -\frac{s_{2\widetilde{12}}}{\widetilde{\Delta m^2}_{32}\widetilde{\Delta m^2}_{21}} & \\ \frac{s_{2\widetilde{12}}}{\widetilde{\Delta m^2}_{31}\widetilde{\Delta m^2}_{21}} & \frac{c^2_{\widetilde{12}}}{(\widetilde{\Delta m^2}_{32})^2} & \\ & & \left[\frac{c^2_{\widetilde{12}}}{(\widetilde{\Delta m^2}_{32})^2} + \frac{s^2_{\widetilde{12}}}{(\widetilde{\Delta m^2}_{31})^2}\right] \end{pmatrix}$$

The Two Matter Angles



Zeroth Order Coefficients

$$P_{\alpha\beta} = \delta_{\alpha\beta} + 4C_{21}^{\alpha\beta}\sin^2\Delta_{21} + 4C_{31}^{\alpha\beta}\sin^2\Delta_{31} + 4C_{32}^{\alpha\beta}\sin^2\Delta_{32} + 8D^{\alpha\beta}\sin\Delta_{21}\sin\Delta_{31}\sin\Delta_{32}$$

$\nu_\alpha \to \nu_\beta$	$(C_{21}^{\alpha\beta})^{(0)}$
$\nu_e \to \nu_e$	$-c_{\widetilde{13}}^4 s_{\widetilde{12}}^2 c_{\widetilde{12}}^2$
$\nu_\mu \to \nu_e$	$c_{\widetilde{13}}^2 s_{\widetilde{12}}^2 c_{\widetilde{12}}^2 (c_{23}^2 - s_{\widetilde{13}}^2 s_{23}^2) + c_{2\widetilde{12}} J_r^m c_\delta$
$\nu_\mu \to \nu_\mu$	$-(c_{23}^2 c_{\widetilde{12}}^2 + s_{23}^2 s_{\widetilde{13}}^2 s_{\widetilde{12}}^2)(c_{23}^2 s_{\widetilde{12}}^2 + s_{23}^2 s_{\widetilde{13}}^2 c_{\widetilde{12}}^2)$ $-2(c_{23}^2 - s_{\widetilde{13}}^2 s_{23}^2) c_{2\widetilde{12}} J_{rr}^m c_\delta + (2J_{rr}^m c_\delta)^2$

$\nu_\alpha \to \nu_\beta$	$(C_{31}^{\alpha\beta})^{(0)}$	$(D^{\alpha\beta})^{(0)}$
$\nu_e \to \nu_e$	$-c_{\widetilde{13}}^2 s_{\widetilde{13}}^2 c_{\widetilde{12}}^2$	0
$\nu_\mu \to \nu_e$	$s_{\widetilde{13}}^2 c_{\widetilde{13}}^2 c_{\widetilde{12}}^2 s_{23}^2 + J_r^m c_\delta$	$-J_r^m s_\delta$
$\nu_\mu \to \nu_\mu$	$-c_{\widetilde{13}}^2 s_{23}^2 (c_{23}^2 s_{\widetilde{12}}^2 + s_{23}^2 s_{\widetilde{13}}^2 c_{\widetilde{12}}^2)$ $-2s_{23}^2 J_r^m c_\delta$	0

$$J_r^m \equiv s_{\widetilde{12}} c_{\widetilde{12}} s_{\widetilde{13}} c_{\widetilde{13}}^2 s_{23} c_{23}, \quad J_{rr}^m \equiv J_r^m / c_{\widetilde{13}}^2$$

Verifying the CPV Term in Matter

The amount of CPV is

$$J \sin\Delta_{21} \sin\Delta_{31} \sin\Delta_{32}$$

where the Jarlskog is

$$J = 8c_{12}s_{12}c_{13}^2 s_{13}c_{23}s_{23}s_\delta$$

C. Jarlskog, PRL 55 (1985)

The exact term in matter is known to be

$$\frac{\widehat{J}}{J} = \frac{\Delta m^2_{21}\Delta m^2_{31}\Delta m^2_{32}}{\Delta\widehat{m^2}_{21}\Delta\widehat{m^2}_{31}\Delta\widehat{m^2}_{32}}$$

V. Naumov, Int. J. Mod. Phys. 1992

P. Harrison, W. Scott, hep-ph/9912435

Our expression reproduces this order by order in ϵ' for all channels.

Angles in Matter

Angles receive corrections at first order:

$$\widetilde{\theta}_{12}^{(1)} = \epsilon' \Delta m^2_{ee} s_{\widetilde{12}} c_{\widetilde{12}} \left(\frac{1}{\widetilde{\Delta m^2}_{32}} - \frac{1}{\widetilde{\Delta m^2}_{31}} \right)$$

$$\widetilde{\theta}_{13}^{(1)} = -\epsilon' \Delta m^2_{ee} \frac{s_{\widetilde{13}}}{c_{\widetilde{13}}} \left(\frac{s^2_{\widetilde{12}}}{\widetilde{\Delta m^2}_{31}} + \frac{c^2_{\widetilde{12}}}{\widetilde{\Delta m^2}_{32}} \right)$$

$$\widetilde{\theta}_{23}^{(1)} = \epsilon' \Delta m^2_{ee} \frac{c_\delta}{c_{\widetilde{13}}} \left(\frac{s^2_{\widetilde{12}}}{\widetilde{\Delta m^2}_{31}} + \frac{c^2_{\widetilde{12}}}{\widetilde{\Delta m^2}_{32}} \right)$$

$$\widetilde{\delta}^{(1)} = -\epsilon' \Delta m^2_{ee} \frac{2 c_{2\widetilde{23}} s_\delta}{s_{2\widetilde{23}} c_{\widetilde{13}}} \left(\frac{s^2_{\widetilde{12}}}{\widetilde{\Delta m^2}_{31}} + \frac{c^2_{\widetilde{12}}}{\widetilde{\Delta m^2}_{32}} \right)$$

Second order: see paper

Other Expressions

We were not the first to examine this problem.

- Madrid: drop $\frac{\Delta m^2_{21}}{\Delta m^2_{31}}$ and s_{13} terms; $\sim |\text{sum of two amplitudes}|^2$

$$P_{\mu e} = 4s^2_{23}s^2_{13}c^2_{13}\left(\frac{\Delta m^2_{31}}{b}\right)^2 \sin^2\Delta_b + 4c^2_{23}s^2_{12}c^2_{12}\left(\frac{\Delta m^2_{21}}{a}\right)^2 \sin^2\Delta_a$$
$$+ 8J_r\frac{\Delta m^2_{21}}{a}\frac{\Delta m^2_{31}}{b}\sin\Delta_a\sin\Delta_b\cos(\delta+\Delta_{31})\ ,\quad b = a - \Delta m^2_{31}$$

A. Cervera, et al., hep-ph/0002108

Other Expressions

We were not the first to examine this problem.

- Madrid: drop $\frac{\Delta m^2_{21}}{\Delta m^2_{31}}$ and s_{13} terms; $\sim$ |sum of two amplitudes|^2

$$P_{\mu e} = 4s^2_{23}s^2_{13}c^2_{13}\left(\frac{\Delta m^2_{31}}{b}\right)^2 \sin^2\Delta_b + 4c^2_{23}s^2_{12}c^2_{12}\left(\frac{\Delta m^2_{21}}{a}\right)^2 \sin^2\Delta_a$$
$$+ 8J_r\frac{\Delta m^2_{21}}{a}\frac{\Delta m^2_{31}}{b}\sin\Delta_a\sin\Delta_b\cos\left(\delta+\Delta_{31}\right)\ , \quad b = a - \Delta m^2_{31}$$

A. Cervera, et al., hep-ph/0002108

E. Akhmedov, et al., hep-ph/0402175

A. Friedland, C. Lunardini, hep-ph/0606101

H. Nunokawa, S. Parke, J. Valle, 0710.0554

Other Expressions

We were not the first to examine this problem.

- Madrid: drop $\frac{\Delta m^2_{21}}{\Delta m^2_{31}}$ and s_{13} terms; $\sim |\text{sum of two amplitudes}|^2$

$$P_{\mu e} = 4 s^2_{23} s^2_{13} c^2_{13} \left(\frac{\Delta m^2_{31}}{b}\right)^2 \sin^2 \Delta_b + 4 c^2_{23} s^2_{12} c^2_{12} \left(\frac{\Delta m^2_{21}}{a}\right)^2 \sin^2 \Delta_a$$
$$+ 8 J_r \frac{\Delta m^2_{21}}{a} \frac{\Delta m^2_{31}}{b} \sin \Delta_a \sin \Delta_b \cos(\delta + \Delta_{31}) \ , \quad b = a - \Delta m^2_{31}$$

A. Cervera, et al., hep-ph/0002108

E. Akhmedov, et al., hep-ph/0402175

A. Friedland, C. Lunardini, hep-ph/0606101

H. Nunokawa, S. Parke, J. Valle, 0710.0554

- AKT: from mass basis rotated 12 then 23 converted into 13
  - Δm^2_{ee} appears all over the expressions

S. Agarwalla, Y. Kao, T. Takeuchi, 1302.6773

Other Expressions

- AM: Powers of $s_{13}^2 \simeq \frac{\Delta m_{21}^2}{\Delta m_{31}^2}$ through the 5/2 order

  K. Asano, H. Minakata, `1103.4387`

- Various other expressions

  J. Arafune, M. Koike, J. Sato, `hep-ph/9703351`

  M. Freund, `hep-ph/0103300`

  E. Akhmedov, et al., `hep-ph/0402175`

Others...

Which is best?

Other Expressions

- AM: Powers of $s_{13}^2 \simeq \frac{\Delta m^2_{21}}{\Delta m^2_{31}}$ through the 5/2 order

K. Asano, H. Minakata, 1103.4387

- Various other expressions

J. Arafune, M. Koike, J. Sato, hep-ph/9703351

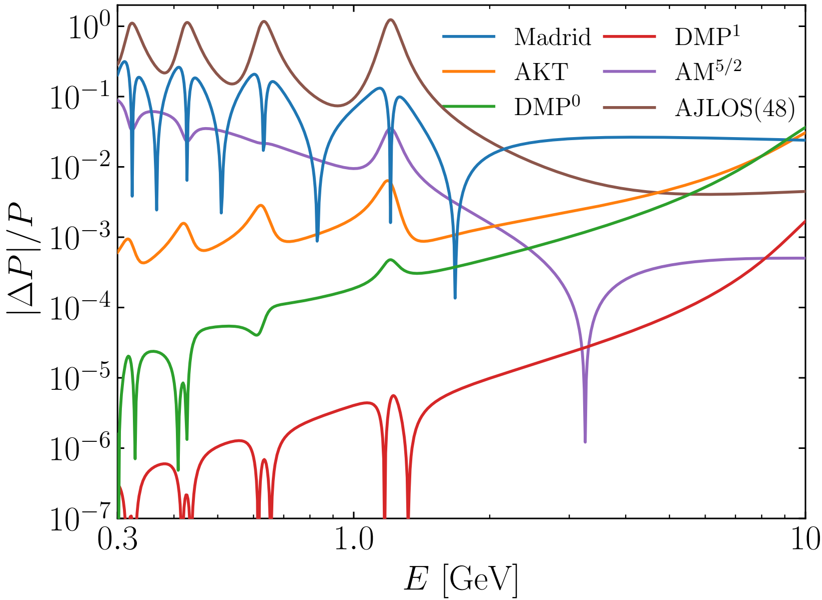
M. Freund, hep-ph/0103300

E. Akhmedov, et al., hep-ph/0402175

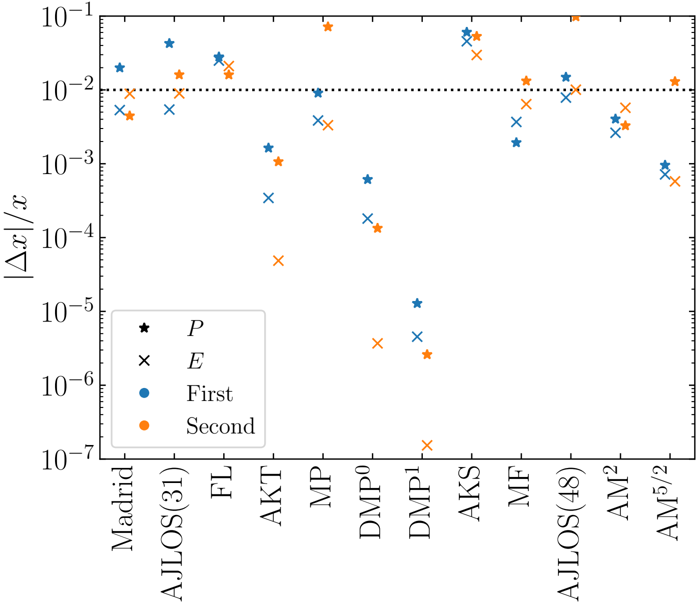
Others...

Which is best?
What does "best" mean?

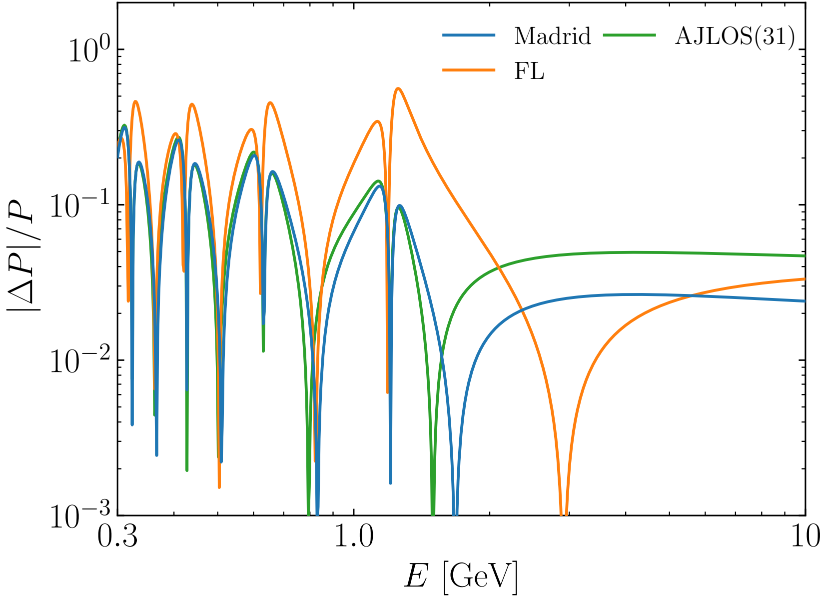
Comparative Precision ($L = 1300$ km)



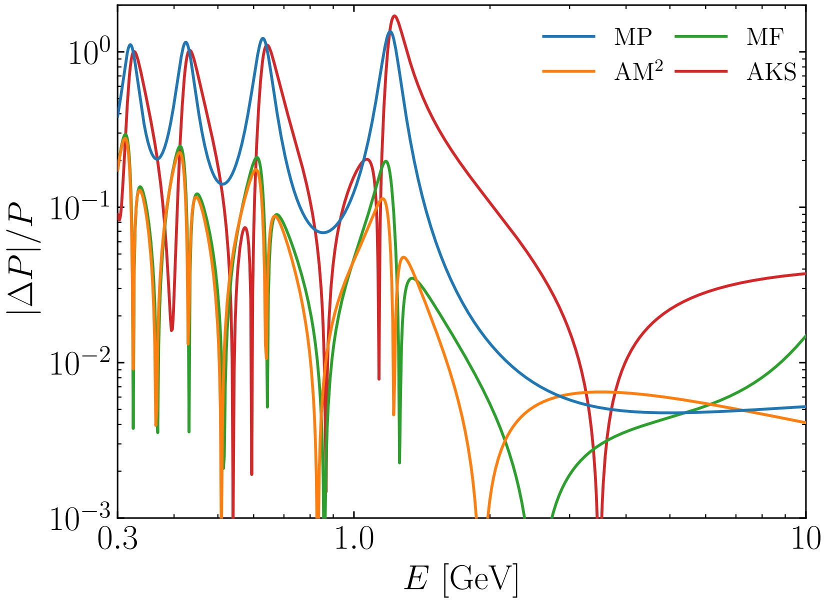
Comparative Precision: At the Peaks



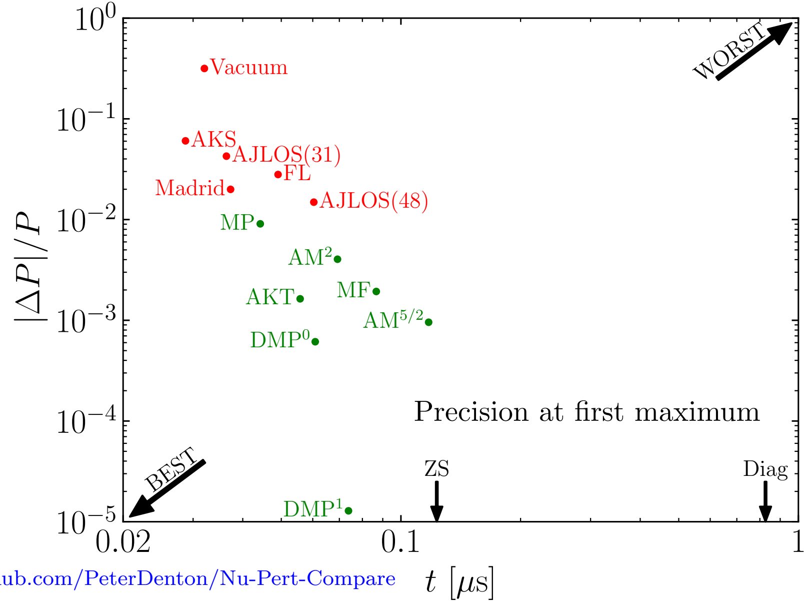
Comparative Precision



Comparative Precision



Speed $\propto$ Simplicity



github.com/PeterDenton/Nu-Pert-Compare

Proper Expansions

Parameter x is an *expansion parameter* iff

$$\lim_{x\to 0} P_{\rm approx}(x) = P_{\rm exact}(x=0)$$

	ϵ	s_{13}	$a/\Delta m^2_{31}$	
Madrid(like)	✗	✗	✗	Cervera+, hep-ph/0002108
AKT	✓	✓	✓	Agarwalla+, 1302.6773
MP	✓	✗	✗	Minakata, Parke, 1505.01826
DMP	✓	✓	✓	PBD+, 1604.08167
AKS	✗	✗	✗	Arafune+, hep-ph/9703351
MF	✓	✗	✗	Freund, hep-ph/0103300
AJLOS(48)	✓	✗	✗	Akhmedov+, hep-ph/0402175
AM	✗	✗	✗	Asano, Minakata, 1103.4387

$$\epsilon \equiv \frac{\Delta m^2_{21}}{\Delta m^2_{ee}}$$

Comparative Review

- Many expressions in the literature (12 considered)
- Most are not at the 1% level
- Most are not exact in vacuum
- Changing the basis to remove level crossings seems best
  - AKT, (MP), DMP
  - Δm^2_{ee} naturally appears (regardless of the name)
- The order of rotations matters:
  - Constant 23 rotation, then in matter: 13, 12
- First order DMP corrections are quite simple