Abstract

As DUNE and T2HK ramp up their efforts, it is a good time to examine what the oscillation probabilities actually are. I will develop a framework for neutrino oscillations in matter that leads to simple and precise expressions. These expressions are sufficiently accurate for current, planned, and proposed oscillation experiments. Further improvements to these expressions can be derived via either changing the basis or perturbation theory, or both. While other expressions exist on the market, I will show how these expressions are significantly more precise and as simple. I will explore how Δm^2_{ee} is modified in matter and how the previous techniques provide a simple and precise expression for this quantity as well. Finally, I will discuss recent results on understanding CP violation in the neutrino sector.

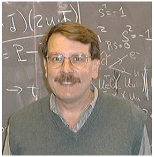
Neutrino Oscillation Probabilities in Matter

Peter B. Denton

UC Irvine

May 22, 2019

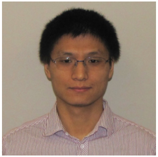
Analytic Oscillation Probability Collaborators



Stephen Parke

Hisakazu Minakata

Gabriela Barenboim



Xining Zhang

Christoph Ternes

1604.08167, 1806.01277, 1808.09453, 1902.00517, 1902.07185

github.com/PeterDenton/Nu-Pert

github.com/PeterDenton/Nu-Pert-Compare

Neutrino Oscillation Parameters Status

Six parameters:

1. $\theta_{13} = (8.6 \pm 0.1)^\circ$
2. $\theta_{12} = (33.8 \pm 0.8)^\circ$
3. $\Delta m^2_{21} = (7.4 \pm 0.2) \times 10^{-5}$ eV^2
4. $\theta_{23} \sim 45^\circ$ (octant)
5. $|\Delta m^2_{31}| = (2.52 \pm 0.03) \times 10^{-3}$ eV^2 (mass ordering)
6. $\delta = ???$

NuFIT, `1811.05487`

PMNS order allows for easy measurement of θ_{13} and θ_{12}.

θ_{23} and $\delta_{\rm CP}$ require full three-flavor description.

Analytic Oscillation Probabilities in Matter

- Solar: $P_{ee} \simeq \sin^2\theta_\odot$

  Approx: S. Mikheev, A. Smirnov, Nuovo Cim. C9 (1986) 17-26

  Exact: S. Parke, PRL 57 (1986) 2322

- Long-baseline: All three flavors

  Exact: H. Zaglauer, K. Schwarzer, Z.Phys. C40 (1988) 273

  Approx: PBD, H. Minakata, S. Parke, `1604.08167`

- ν_e disappearance (nu storm):
  $\Delta\widehat{m^2}_{ee} = \widehat{m^2}_3 - (\widehat{m^2}_1 + \widehat{m^2}_2 - \Delta m^2_{21} c^2_{12})$

  PBD, S. Parke, `1808.09453`

- Atmospheric?

What is $P(\nu_\mu \to \nu_e)$?

$$P(\overset{(-)}{\nu}_\mu \to \overset{(-)}{\nu}_e) = |\mathcal{A}_{\mu e}|^2 \qquad \mathcal{A}_{\mu e} = \mathcal{A}_{31} + e^{\pm i\Delta_{32}}\mathcal{A}_{21}$$

$$\mathcal{A}_{31} = 2s_{13}c_{13}s_{23}\sin\Delta_{31}$$

$$\mathcal{A}_{21} = 2s_{12}c_{13}(c_{12}c_{23}e^{i\delta} - s_{12}s_{13}s_{23})\sin\Delta_{21}$$

$$\Delta_{ij} = \Delta m^2{}_{ij}L/4E$$

The Several Billion Dollar Question

What is $P(\nu_\mu \to \nu_e)$?

$$P(\overset{(-)}{\nu}_\mu \to \overset{(-)}{\nu}_e) = |\mathcal{A}_{\mu e}|^2 \qquad \mathcal{A}_{\mu e} = \mathcal{A}_{31} + e^{\pm i\Delta_{32}}\mathcal{A}_{21}$$

$$\mathcal{A}_{31} = 2s_{13}c_{13}s_{23}\sin\Delta_{31}$$

$$\mathcal{A}_{21} = 2s_{12}c_{13}(c_{12}c_{23}e^{i\delta} - s_{12}s_{13}s_{23})\sin\Delta_{21}$$

$$\Delta_{ij} = \Delta m^2{}_{ij}L/4E$$

...in matter?

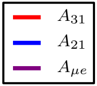
Now: NOvA, T2K, MINOS, ...
Upcoming: DUNE, T2HK, ...
Second maximum: T2HKK? ESSnuSB? ...

Neutrino oscillations in vacuum: appearance

$\delta = 0.0\pi$

$\Delta_{32} = 0.5\pi$

NO



$$P(\nu_\mu \to \nu_e) = A_{\mu e} A_{\mu e}^*$$

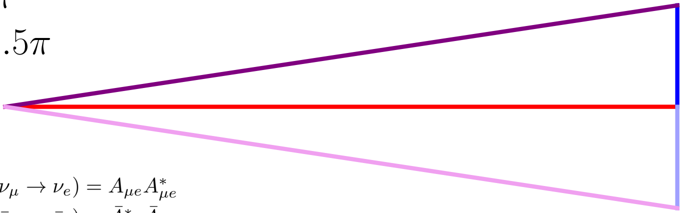
Denton & Parke

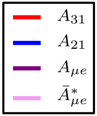
Neutrino oscillations in vacuum: appearance

$\delta = 0.0\pi$

$\Delta_{32} = 0.5\pi$

NO





$P(\nu_\mu \to \nu_e) = A_{\mu e} A^*_{\mu e}$

$P(\bar{\nu}_\mu \to \bar{\nu}_e) = \bar{A}^*_{\mu e} \bar{A}_{\mu e}$

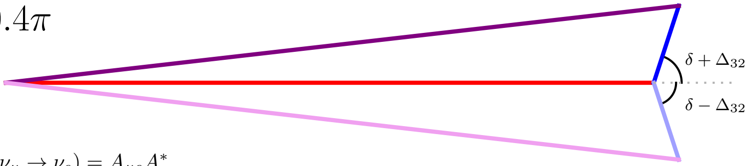
Denton & Parke

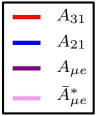
Neutrino oscillations in vacuum: appearance

$\delta = 0.0\pi$

$\Delta_{32} = 0.4\pi$

NO





$P(\nu_\mu \to \nu_e) = A_{\mu e} A^*_{\mu e}$

$P(\bar{\nu}_\mu \to \bar{\nu}_e) = \bar{A}^*_{\mu e} \bar{A}_{\mu e}$

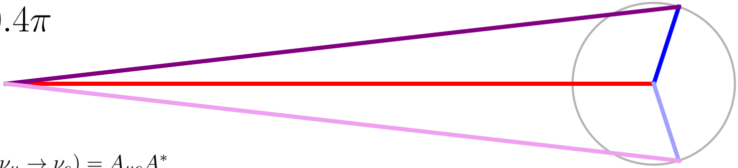
Denton & Parke

Neutrino oscillations in vacuum: appearance

$\delta = 0.0\pi$

$\Delta_{32} = 0.4\pi$

NO



$$P(\nu_\mu \to \nu_e) = A_{\mu e} A^*_{\mu e}$$
$$P(\bar{\nu}_\mu \to \bar{\nu}_e) = \bar{A}^*_{\mu e} \bar{A}_{\mu e}$$

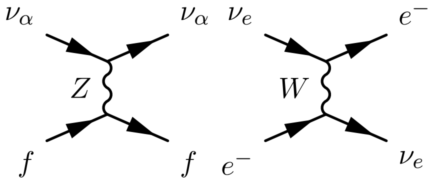
Denton & Parke

Matter Effects Matter

Call Schrödinger equation's eigenvalues m_i^2 and eigenvectors U_i.

$$\mathcal{A}(\nu_\alpha \to \nu_\beta) = \sum_{i=1}^{3} U_{\alpha i}^* U_{\beta i} e^{-i m_i^2 L/2E} \qquad P = |\mathcal{A}|^2$$

In matter ν's propagate in a new basis that depends on $a \propto \rho E$.



L. Wolfenstein, PRD 17 (1978)

Eigenvalues: $m_i^2 \to \widehat{m^2}_i(a)$

Eigenvectors are given by $\theta_{ij} \to \widehat{\theta}_{ij}(a) \qquad \Leftarrow \qquad$ Unitarity

Variable Matter Density

We assume ρ is constant. Is this okay?

If ρ varies only "slowly," we can set ρ to the average:

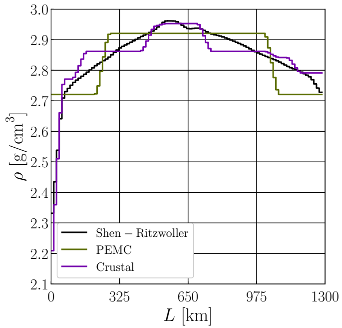
$$\rho(x) \to \bar{\rho} = \frac{1}{L}\int_0^L \rho(x)dx$$

ρ doesn't vary "too much" when

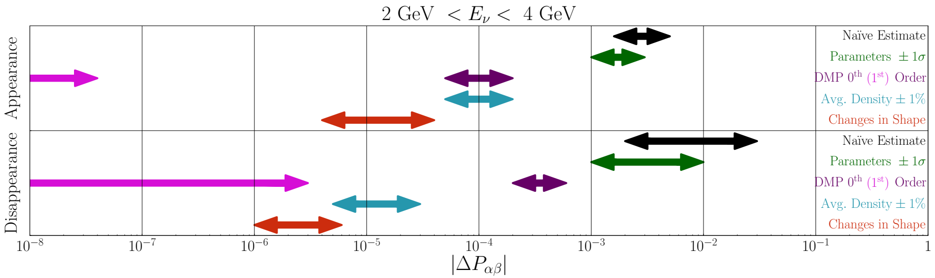
$$\left|\frac{d\widehat{\theta}}{dt}\right| \ll \left|\frac{\widehat{\Delta m^2}}{2E}\right|$$

True for DUNE?

Variable Matter Density



This is a great approximation at DUNE: ✓!



K. Kelly, S. Parke, 1802.06784

Hamiltonian Dynamics

$$H=\frac{1}{2E}\left[U\begin{pmatrix}0&&\\&\Delta m^2_{21}&\\&&\Delta m^2_{31}\end{pmatrix}U^\dagger+\begin{pmatrix}a&&\\&0&\\&&0\end{pmatrix}\right]$$

$a=2\sqrt{2}G_FN_eE$

$$U=\begin{pmatrix}1&&\\&c_{23}&s_{23}\\&-s_{23}&c_{23}\end{pmatrix}\begin{pmatrix}c_{13}&&s_{13}e^{-i\delta}\\&1&\\-s_{13}e^{i\delta}&&c_{13}\end{pmatrix}\begin{pmatrix}c_{12}&s_{12}&\\-s_{12}&c_{12}&\\&&1\end{pmatrix}$$

Find eigenvalues and eigenvectors:

$$H=\frac{1}{2E}\widehat{U}\begin{pmatrix}0&&\\&\widehat{\Delta m^2}_{21}&\\&&\widehat{\Delta m^2}_{31}\end{pmatrix}\widehat{U}^\dagger$$

Computationally works, but we can do better than a $\boxed{\textbf{black box}}$...

Eigenvalues Analytically: The Exact Solution

Solve the cubic characteristic equation.

$$\widehat{m^2}_1 = \frac{A}{3} - \frac{1}{3}\sqrt{A^2-3B}\,S - \frac{\sqrt{3}}{3}\sqrt{A^2-3B}\sqrt{1-S^2}$$

$$\widehat{m^2}_2 = \frac{A}{3} - \frac{1}{3}\sqrt{A^2-3B}\,S + \frac{\sqrt{3}}{3}\sqrt{A^2-3B}\sqrt{1-S^2}$$

$$\widehat{m^2}_3 = \frac{A}{3} + \frac{2}{3}\sqrt{A^2-3B}\,S$$

$$A = \Delta m^2_{21} + \Delta m^2_{31} + a$$

$$B = \Delta m^2_{21}\Delta m^2_{31} + a\left[c^2_{13}\Delta m^2_{31} + (c^2_{12}c^2_{13} + s^2_{13})\Delta m^2_{21}\right]$$

$$C = a\Delta m^2_{21}\Delta m^2_{31}c^2_{12}c^2_{13}$$

$$S = \cos\left\{\frac{1}{3}\cos^{-1}\left[\frac{2A^3 - 9AB + 27C}{2\left(A^2-3B\right)^{3/2}}\right]\right\}$$

H. Zaglauer, K. Schwarzer, Z.Phys. C40 (1988) 273

Traded one $\boxed{\textbf{black box}}$ for another...

We're physicists so ...

Perturbation theory

Alternative Solutions

Perturbative expansion:

- Small matter potential: $a/\Delta m^2$

Y. Li, Y. Wang, Z-z. Xing, 1605.00900

- s_{13}, s_{13}^2

A. Cervera, et al., hep-ph/0002108

H. Minakata, 0910.5545

K. Asano, H. Minakata, 1103.4387

- $\Delta m^2_{21}/\Delta m^2_{31} \sim 0.03$

J. Arafune, J. Sato, hep-ph/9607437

A. Cervera, et al., hep-ph/0002108

M. Freund, hep-ph/0103300

E. Akhmedov, et al., hep-ph/0402175

M. Blennow, A. Smirnov, 1306.2903

H. Minakata, S. Parke, 1505.01826

PBD, H. Minakata, S. Parke, 1604.08167

Alternative Solutions: Example

$$P_0 = \sin^2\theta_{23} \frac{\sin^2 2\theta_{13}}{\hat{C}^2} \sin^2(\hat{\Delta}\hat{C}), \tag{36a}$$

$$P_{\sin\delta} = \frac{1}{2}\alpha \frac{\sin\delta\cos\theta_{13}\sin 2\theta_{12}\sin 2\theta_{13}\sin 2\theta_{23}}{\hat{A}\hat{C}\cos\theta_{13}^2}\sin(\hat{C}\hat{\Delta})$$

$$\times\{\cos(\hat{C}\hat{\Delta}) - \cos((1+\hat{A})\hat{\Delta})\}, \tag{36b}$$

$$P_{\cos\delta} = \frac{1}{2}\alpha \frac{\cos\delta\cos\theta_{13}\sin 2\theta_{12}\sin 2\theta_{13}\sin 2\theta_{23}}{\hat{A}\hat{C}\cos\theta_{13}^2}\sin(\hat{C}\hat{\Delta})$$

$$\times\{\sin((1+\hat{A})\hat{\Delta}) \mp \sin(\hat{C}\hat{\Delta})\}, \tag{36c}$$

$$P_1 = -\alpha \frac{1-\hat{A}\cos 2\theta_{13}}{\hat{C}^3}\sin^2\theta_{12}\sin^2 2\theta_{13}\sin^2\theta_{23}\hat{\Delta}$$

$$\times\sin(2\hat{\Delta}\hat{C}) + \alpha\frac{2\hat{A}(-\hat{A}+\cos 2\theta_{13})}{\hat{C}^4}$$

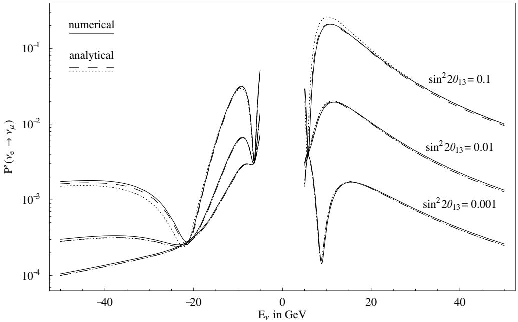
$$\times\sin^2\theta_{12}\sin^2 2\theta_{13}\sin^2\theta_{23}\sin^2(\hat{\Delta}\hat{C}), \tag{36d}$$

$$P_2 = \alpha\frac{\mp 1 + \hat{C} \pm \hat{A}\cos 2\theta_{13}}{2\hat{C}^2\hat{A}\cos^2\theta_{13}}\cos\theta_{13}\sin 2\theta_{12}\sin 2\theta_{13}$$

$$\times\sin 2\theta_{23}\sin^2(\hat{\Delta}\hat{C}), \tag{36e}$$

$$P_3 = \alpha^2\frac{2\hat{C}\cos^2\theta_{23}\sin^2 2\theta_{12}}{\hat{A}^2\cos^2\theta_{13}(\mp\hat{A}+\hat{C}\pm\cos 2\theta_{13})}$$

$$\times\sin^2\left(\frac{1}{2}(1+\hat{A}\mp\hat{C})\hat{\Delta}\right). \tag{36f}$$



M. Freund, hep-ph/0103300

A Tale of Two Tools

Split the Hamiltonian into:

- Large, diagonal part (H_0)
- Small, *off-diagonal* part (H_1)

- Improves precision at zeroth order
- Naturally leads to using $\Delta m^2_{ee} \equiv c^2_{12}\Delta m^2_{31} + s^2_{12}\Delta m^2_{32}$

H. Nunokawa, S. Parke, R. Zukanovich, hep-ph/0503283

1. **Rotations**:
   - A two-flavor rotation only requires solving a quadratic
   - Diagonalize away the big off-diagonal terms
   - Follows the order of the PMNS matrix
2. **Perturbative expansion**:
   - Smallness parameter is $|\epsilon'| \leq 0.015$
   - Correct eigenvalues ($\widetilde{m^2}_i$) and eigenvectors ($\widetilde{\theta_{ij}}$)
   - Eigenvalues already include 1$^{\text{st}}$ order corrections at 0$^{\text{th}}$ order
   - Can improve the precision to arbitrary order

"What is Δm^2_{ee}?"

$$\Delta m^2_{ee} = c^2_{12}\Delta m^2_{31} + s^2_{12}\Delta m^2_{32} = \Delta m^2_{31} - s^2_{12}\Delta m^2_{21}$$

H. Nunokawa, S. Parke, R. Funchal, hep-ph/0503283

S. Parke, 1601.07464

Additional expressions for $\Delta m^2_{\mu\mu}, \Delta m^2_{\tau\tau}$

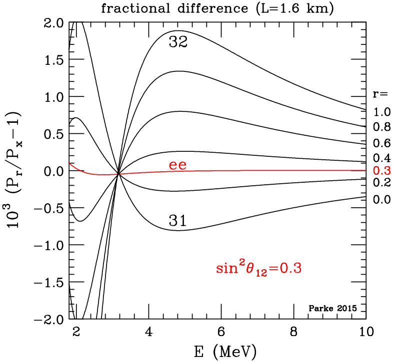
Useful definitions:

- ν_e weighted average of atmospheric splittings:

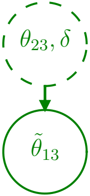
$$m_3^2 - \frac{|U_{e1}|^2 m_1^2 + |U_{e2}|^2 m_2^2}{|U_{e1}|^2 + |U_{e2}|^2}$$

- Measured by reactor experiments with smallest L/E error
- Simple form:
  $P_{ee} \approx 1 - \sin^2 2\theta_{13} \sin^2 \Delta_{ee}$

$$\Delta_{ij} = \Delta m^2{}_{ij} L/4E$$

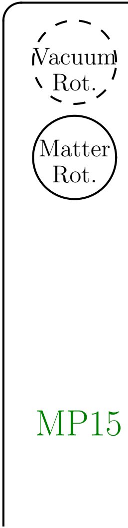


Atmospheric Resonance

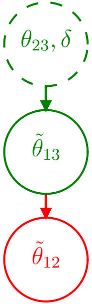


1. $U_{23}(\theta_{23}, \delta)$ commutes with matter potential
2. Largest off-diagonal term:
   $s_{13}c_{13}\Delta m^2_{ee}$ in the 1-3 position

- Eigenvalues still cross at the solar resonance:
  - No perturbation theory there
- Smallness parameter:
  - After U_{23}: $s_{13}c_{13} = 0.15$
  - After U_{13}: $s_{12}c_{12}\frac{\Delta m^2_{21}}{\Delta m^2_{ee}} = 0.015$



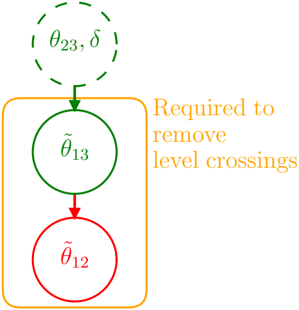
Solar Resonance



3. Largest off-diagonal term:
   $s_{12}c_{12}c_{\widetilde{\theta}_{13}-\theta_{13}}\Delta m^2_{21}$ in the 1-2 position
   - Largest except for ν's above the atmospheric resonance

- $|\epsilon'| < 0.015$, zero in vacuum
- Perturbation theory valid everywhere now
- Rotation order matches PMNS
- Take vacuum expressions, replace θ_{13}, θ_{12}, and Δm^2_{ij}
- Extremely precise $|\Delta P/P| < 10^{-3}$

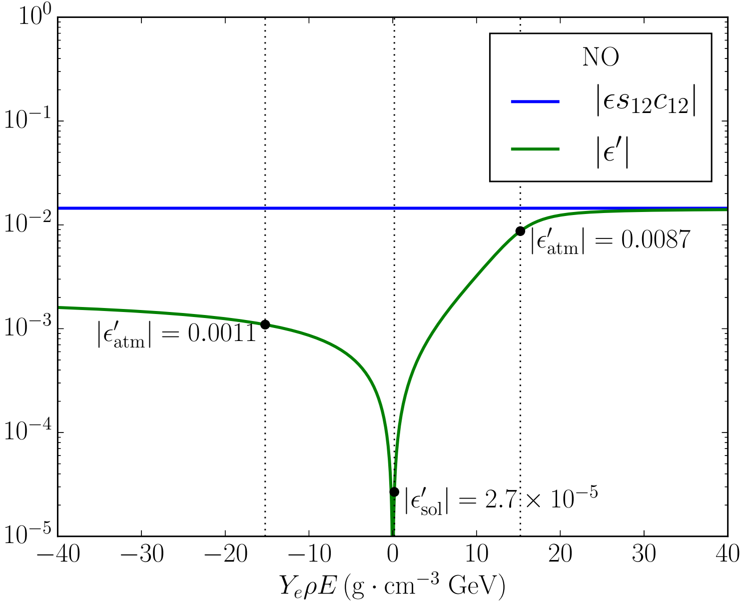
Solar Resonance



3. Largest off-diagonal term:
   $s_{12}c_{12}c_{\widetilde{\theta}_{13}-\theta_{13}}\Delta m^2_{21}$ in the 1-2 position
   - Largest except for ν's above the atmospheric resonance

- $|\epsilon'| < 0.015$, zero in vacuum
- Perturbation theory valid everywhere now
- Rotation order matches PMNS
- Take vacuum expressions, replace θ_{13}, θ_{12}, and Δm^2_{ij}
- Extremely precise $|\Delta P/P| < 10^{-3}$

Expansion Parameter



Probability in Matter: DMP 0^{th}

Matter expression $\Rightarrow$ Vacuum expression

$$\widetilde{P}_{\alpha\beta}(\Delta m^2_{21}, \Delta m^2_{31}, \theta_{13}, \theta_{12}, \theta_{23}, \delta) = P_{\alpha\beta}(\Delta\widetilde{m^2}_{21}, \Delta\widetilde{m^2}_{31}, \widetilde{\theta}_{13}, \widetilde{\theta}_{12}, \theta_{23}, \delta)$$

Same expression, 4 new variables.

Probability in Matter: DMP 0^{th}

Matter expression $\Rightarrow$ Vacuum expression

$$\widetilde{P}_{\alpha\beta}(\Delta m^2_{21}, \Delta m^2_{31}, \theta_{13}, \theta_{12}, \theta_{23}, \delta) = P_{\alpha\beta}(\Delta\widetilde{m^2}_{21}, \Delta\widetilde{m^2}_{31}, \widetilde{\theta}_{13}, \widetilde{\theta}_{12}, \theta_{23}, \delta)$$

Same expression, 4 new variables.

$$\cos 2\widetilde{\theta}_{13} = \frac{\Delta m^2_{ee}\cos 2\theta_{13} - a}{\Delta\widetilde{m^2}_{ee}}$$

$$\Delta\widetilde{m^2}_{ee} = \Delta m^2_{ee}\sqrt{(\cos 2\theta_{13} - a/\Delta m^2_{ee})^2 + \sin^2 2\theta_{13}}$$

Probability in Matter: DMP 0$^{\text{th}}$

Matter expression $\Rightarrow$ Vacuum expression

$$\widetilde{P}_{\alpha\beta}(\Delta m^2_{21}, \Delta m^2_{31}, \theta_{13}, \theta_{12}, \theta_{23}, \delta) = P_{\alpha\beta}(\Delta\widetilde{m^2}_{21}, \Delta\widetilde{m^2}_{31}, \widetilde{\theta}_{13}, \widetilde{\theta}_{12}, \theta_{23}, \delta)$$

Same expression, 4 new variables.

$$\cos 2\widetilde{\theta}_{13} = \frac{\Delta m^2_{ee}\cos 2\theta_{13} - a}{\Delta\widetilde{m^2}_{ee}}$$

$$\Delta\widetilde{m^2}_{ee} = \Delta m^2_{ee}\sqrt{(\cos 2\theta_{13} - a/\Delta m^2_{ee})^2 + \sin^2 2\theta_{13}}$$

$$\cos 2\widetilde{\theta}_{12} = \frac{\Delta m^2_{21}\cos 2\theta_{12} - a_{12}}{\Delta\widetilde{m^2}_{21}}\,, \qquad a_{12} = (a + \Delta m^2_{ee} - \Delta\widetilde{m^2}_{ee})/2$$

$$\Delta\widetilde{m^2}_{21} = \Delta m^2_{21}\sqrt{(\cos 2\theta_{12} - a_{12}/\Delta m^2_{21})^2 + \cos^2(\widetilde{\theta}_{13} - \theta_{13})\sin^2 2\theta_{12}}$$

Probability in Matter: DMP 0th

Matter expression $\Rightarrow$ Vacuum expression

$$\widetilde{P}_{\alpha\beta}(\Delta m^2_{21}, \Delta m^2_{31}, \theta_{13}, \theta_{12}, \theta_{23}, \delta) = P_{\alpha\beta}(\Delta\widetilde{m^2}_{21}, \Delta\widetilde{m^2}_{31}, \widetilde{\theta}_{13}, \widetilde{\theta}_{12}, \theta_{23}, \delta)$$

Same expression, 4 new variables.

$$\cos 2\widetilde{\theta}_{13} = \frac{\Delta m^2_{ee}\cos 2\theta_{13} - a}{\Delta\widetilde{m^2}_{ee}}$$

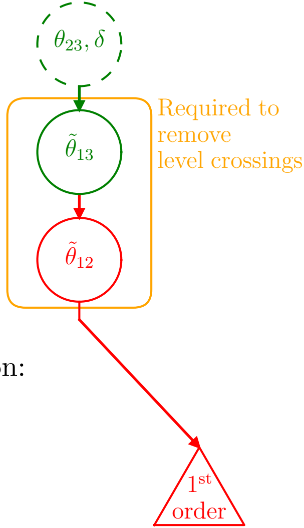
$$\Delta\widetilde{m^2}_{ee} = \Delta m^2_{ee}\sqrt{(\cos 2\theta_{13} - a/\Delta m^2_{ee})^2 + \sin^2 2\theta_{13}}$$

$$\cos 2\widetilde{\theta}_{12} = \frac{\Delta m^2_{21}\cos 2\theta_{12} - a_{12}}{\Delta\widetilde{m^2}_{21}}, \qquad a_{12} = (a + \Delta m^2_{ee} - \Delta\widetilde{m^2}_{ee})/2$$

$$\Delta\widetilde{m^2}_{21} = \Delta m^2_{21}\sqrt{(\cos 2\theta_{12} - a_{12}/\Delta m^2_{21})^2 + \cos^2(\widetilde{\theta}_{13} - \theta_{13})\sin^2 2\theta_{12}}$$

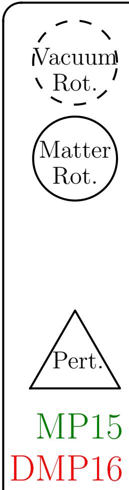
$$\Delta\widetilde{m^2}_{31} = \Delta m^2_{31} + \frac{1}{4}a + \frac{1}{2}(\Delta\widetilde{m^2}_{21} - \Delta m^2_{21}) + \frac{3}{4}(\Delta\widetilde{m^2}_{ee} - \Delta m^2_{ee})$$

Improve with Perturbation



4. $\gtrsim 2$ orders of magnitude of improvement in precision: $|\Delta P/P| < 10^{-6}$

- Eigenvalues need no correction
- Compact form utilizes a $\widetilde{m^2}_1 \leftrightarrow \widetilde{m^2}_2$, $\widetilde\theta_{12} \leftrightarrow \widetilde\theta_{12} \pm \pi/2$ symmetry



$\widetilde{m^2}_{1,2} - \widetilde{\theta}_{12}$ Symmetry

From the shape of $U_{12}(\widetilde{\theta}_{12})$, it is clear that the probabilities are invariant under a simultaneous interchange of

$$\widetilde{m^2}_1 \leftrightarrow \widetilde{m^2}_2\,, \qquad \text{and} \qquad \widetilde{\theta}_{12} \to \widetilde{\theta}_{12} \pm \frac{\pi}{2}\,.$$

Since only even powers of $\widetilde{\theta}_{12}$ trig functions $c^2_{\widetilde{12}}, s^2_{\widetilde{12}}, c_{\widetilde{12}}s_{\widetilde{12}}, \cos(2\widetilde{\theta}_{12}), \sin(2\widetilde{\theta}_{12})$ appear in the probabilities, the sign degeneracy is irrelevant.

More usefully, we can write that the probabilities are invariant under the simultaneous interchange of

$$\widetilde{m^2}_1 \leftrightarrow \widetilde{m^2}_2\,, \qquad c^2_{\widetilde{12}} \leftrightarrow s^2_{\widetilde{12}}\,, \qquad \text{and} \qquad c_{\widetilde{12}}s_{\widetilde{12}} \to -c_{\widetilde{12}}s_{\widetilde{12}}\,.$$

This interchange constrains the $\sin^2\Delta_{21}$ term, and the $\sin^2\Delta_{32}$ term easily follows from the $\sin^2\Delta_{31}$ term.

General Form of the First Order Coefficients

$$P_{\alpha\beta} = \delta_{\alpha\beta} + 4C_{21}^{\alpha\beta}\sin^2\Delta_{21} + 4C_{31}^{\alpha\beta}\sin^2\Delta_{31} + 4C_{32}^{\alpha\beta}\sin^2\Delta_{32} + 8D^{\alpha\beta}\sin\Delta_{21}\sin\Delta_{31}\sin\Delta_{32}$$

Can reduce 8 expressions down to 3:

$$(C_{21}^{\alpha\beta})^{(1)} = \epsilon'\Delta m^2_{ee}\left(\frac{F_1^{\alpha\beta}}{\widetilde{\Delta m^2}_{31}} + \frac{F_2^{\alpha\beta}}{\widetilde{\Delta m^2}_{32}}\right)$$

$$(C_{31}^{\alpha\beta})^{(1)} = \epsilon'\Delta m^2_{ee}\left(\frac{F_1^{\alpha\beta} + G_1^{\alpha\beta}}{\widetilde{\Delta m^2}_{31}} - \frac{F_2^{\alpha\beta}}{\widetilde{\Delta m^2}_{32}}\right)$$

$$(C_{32}^{\alpha\beta})^{(1)} = \epsilon'\Delta m^2_{ee}\left(-\frac{F_1^{\alpha\beta}}{\widetilde{\Delta m^2}_{31}} + \frac{F_2^{\alpha\beta} + G_2^{\alpha\beta}}{\widetilde{\Delta m^2}_{32}}\right)$$

$$(D^{\alpha\beta})^{(1)} = \epsilon'\Delta m^2_{ee}\left(\frac{K_1^{\alpha\beta}}{\widetilde{\Delta m^2}_{31}} - \frac{K_2^{\alpha\beta}}{\widetilde{\Delta m^2}_{32}}\right)$$

$$K_1^{\alpha\beta} = \mp s_{23}c_{23}c_{\widetilde{13}}s^2_{\widetilde{12}}(c^2_{\widetilde{13}}c^2_{\widetilde{12}} - s^2_{\widetilde{13}})s_\delta\,, \quad \alpha \neq \beta$$

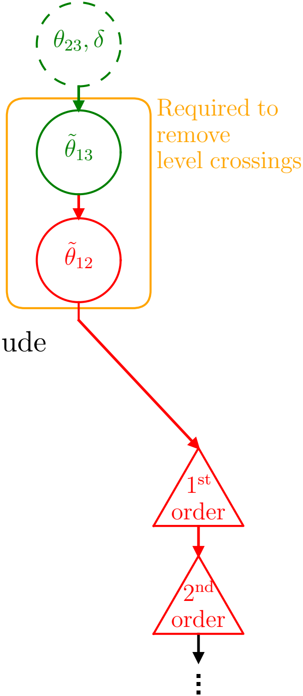
First Order Coefficients

$\nu_\alpha \to \nu_\beta$	$F_1^{\alpha\beta}$
$\nu_e \to \nu_e$	$-2c_{\widetilde{13}}^3 s_{\widetilde{13}} s_{\widetilde{12}}^3 c_{\widetilde{12}}$
$\nu_\mu \to \nu_e$	$c_{\widetilde{13}} s_{\widetilde{12}}^2 [s_{\widetilde{13}} s_{\widetilde{12}} c_{\widetilde{12}} (c_{23}^2 + c_{2\widetilde{13}} s_{23}^2)$ $-s_{23} c_{23} (s_{\widetilde{13}}^2 s_{\widetilde{12}}^2 + c_{2\widetilde{13}} c_{\widetilde{12}}^2) c_\delta]$
$\nu_\mu \to \nu_\mu$	$2c_{\widetilde{13}} s_{\widetilde{12}} (s_{23}^2 s_{\widetilde{13}} c_{\widetilde{12}} + s_{23} c_{23} s_{\widetilde{12}} c_\delta) \times$ $(c_{23}^2 c_{\widetilde{12}}^2 - 2 s_{23} c_{23} s_{\widetilde{13}} s_{\widetilde{12}} c_{\widetilde{12}} c_\delta + s_{23}^2 s_{\widetilde{13}}^2 s_{\widetilde{12}}^2)$

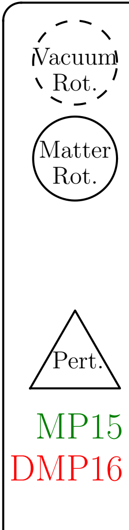
$\nu_\alpha \to \nu_\beta$	$G_1^{\alpha\beta}$
$\nu_e \to \nu_e$	$2s_{\widetilde{13}} c_{\widetilde{13}} s_{\widetilde{12}} c_{\widetilde{12}} c_{2\widetilde{13}}$
$\nu_\mu \to \nu_e$	$-2s_{\widetilde{13}} c_{\widetilde{13}} s_{\widetilde{12}} (s_{23}^2 c_{2\widetilde{13}} c_{\widetilde{12}} - s_{23} c_{23} s_{\widetilde{13}} s_{\widetilde{12}} c_\delta)$
$\nu_\mu \to \nu_\mu$	$-2c_{\widetilde{13}} s_{\widetilde{12}} (s_{23}^2 s_{\widetilde{13}} c_{\widetilde{12}} + s_{23} c_{23} s_{\widetilde{12}} c_\delta)$ $\times(1 - 2c_{\widetilde{13}}^2 s_{23}^2)$

Three channels gives them all with unitarity!

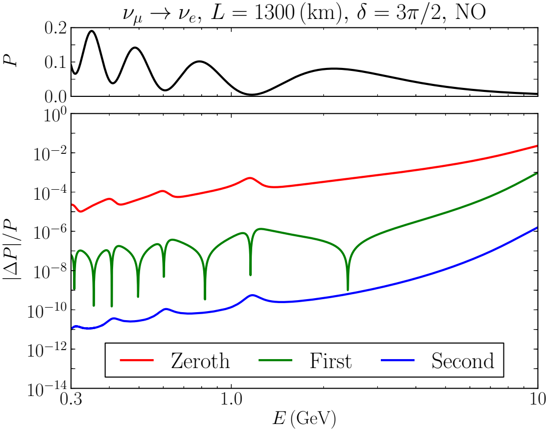
Higher Orders



5. $\gtrsim$ 2 more orders of magnitude of improvement per order: $|\Delta P/P|<10^{-9}$, ...



Precision



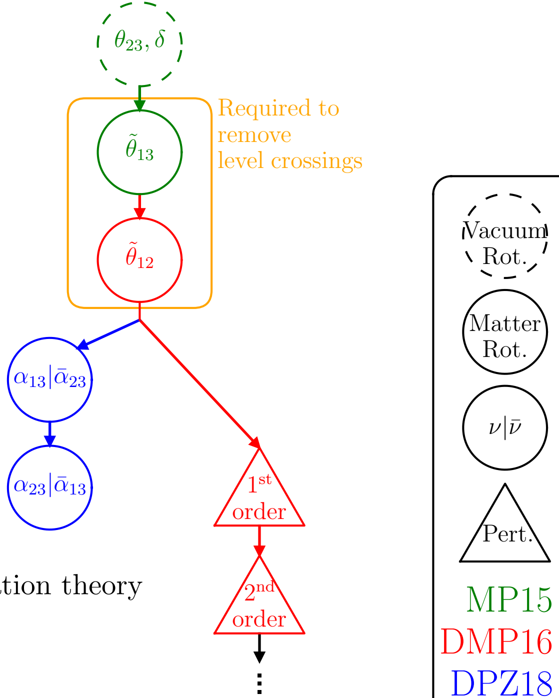
DUNE: NO, $\delta = 3\pi/2$		First min	First max
$P(\nu_\mu \to \nu_e)$		0.0047	0.081
E (GeV)		1.2	2.2
$\frac{\lvert\Delta P\rvert}{P}$	Zeroth	5×10^{-4}	4×10^{-4}
	First	3×10^{-7}	2×10^{-7}
	Second	6×10^{-10}	5×10^{-10}

More Rotations

Instead continue to diagonalize large terms

4. 1-3 sector for ν's
   2-3 sector for $\bar{\nu}$'s
5. Then opposite

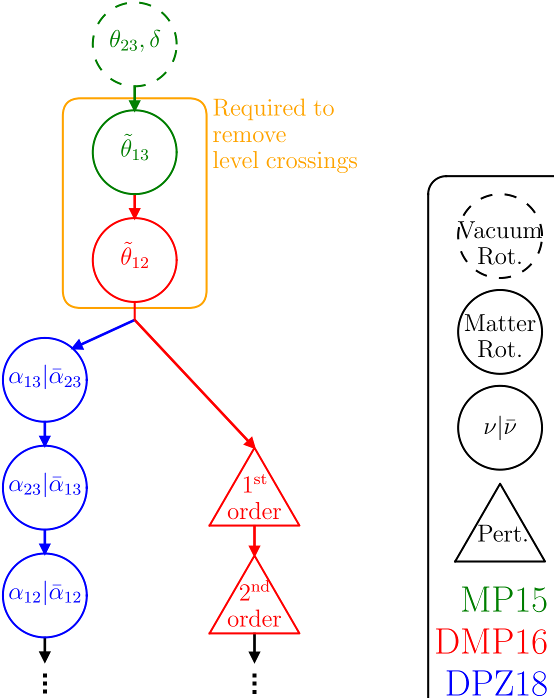
- 2 additional rotations
  $\equiv$ 1 order of perturbation theory



Even More Rotations

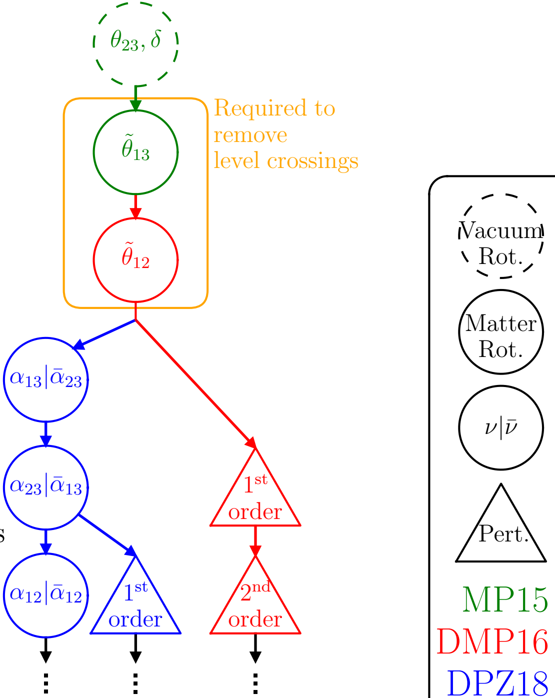
6. 1-2 sector for either $\nu/\bar{\nu}$

- 3 additional rotations
  $\equiv$ 2 orders of pert. th.



More Options

6. Perturbation theory after 2 additional rotations



- 2 additional rotations + 1 order of pert. th. ≡ 2 orders of pert. th.
- The precision of rotations follows Fibonacci sequence

Rotation Precision Follows the Fibonacci Sequence

Given $H = H_0 + H_1$, perform successive rotations,

Rotation #	0	1	2	3	4	5	6	$7 \cdots$
Size of correction	1	1	2	3	5	8	13	$21 \cdots$

Necessary conditions:

1. H_1 is Hermitian
2. H_1 has no diagonal entries
3. At least one off-diagonal entry is zero

Each additional rotation is of uniform complexity,
perturbation theory is of increasing complexity

Verifying the CPV Term in Matter

The amount of CPV is

$$J \sin\Delta_{21} \sin\Delta_{31} \sin\Delta_{32}$$

where the Jarlskog is

$$J = 8c_{12}s_{12}c_{13}^2s_{13}c_{23}s_{23}s_\delta$$

C. Jarlskog, PRL 55 (1985)

The exact term in matter is known to be

$$\frac{\widehat{J}}{J} = \frac{\Delta m^2_{21}\Delta m^2_{31}\Delta m^2_{32}}{\Delta\widehat{m^2}_{21}\Delta\widehat{m^2}_{31}\Delta\widehat{m^2}_{32}}$$

V. Naumov, Int. J. Mod. Phys. 1992

P. Harrison, W. Scott, hep-ph/9912435

Our expression reproduces this order by order in ϵ' for all channels.

CPV in Matter

CPV in matter can be written sans $\cos(\frac{1}{3}\cos^{-1}(\cdots))$ term.

$$\frac{\widehat{J}}{J} = \frac{\Delta m^2_{21}\Delta m^2_{31}\Delta m^2_{32}}{\Delta\widehat{m^2}_{21}\Delta\widehat{m^2}_{31}\Delta\widehat{m^2}_{32}}$$

$$\left(\Delta\widehat{m^2}_{21}\Delta\widehat{m^2}_{31}\Delta\widehat{m^2}_{32}\right)^2 = (A^2-4B)(B^2-4AC)+(2AB-27C)C$$

$$A \equiv \sum_j \widehat{m^2}_j = \Delta m^2_{31} + \Delta m^2_{21} + a$$

$$B \equiv \sum_{j>k} \widehat{m^2}_j\widehat{m^2}_k = \Delta m^2_{31}\Delta m^2_{21} + a(\Delta m^2_{ee}c^2_{13} + \Delta m^2_{21})$$

$$C \equiv \prod_j \widehat{m^2}_j = a\Delta m^2_{31}\Delta m^2_{21}c^2_{13}c^2_{12}$$

This is the *only* oscillation quantity in matter that can be written exactly without $\cos(\frac{1}{3}\cos^{-1}(\cdots))$!

CPV in Matter

Thus $\widehat{J}^2$ is fourth order in matter potential:
only two matter corrections are really needed.

CPV in Matter

Thus $\widehat{J}^2$ is fourth order in matter potential:
only two matter corrections are really needed.

CPV in matter can be approximated:

$$\frac{\widehat{J}}{J} \approx \frac{1}{\mathcal{S}_{\rm atm}\mathcal{S}_\odot}$$

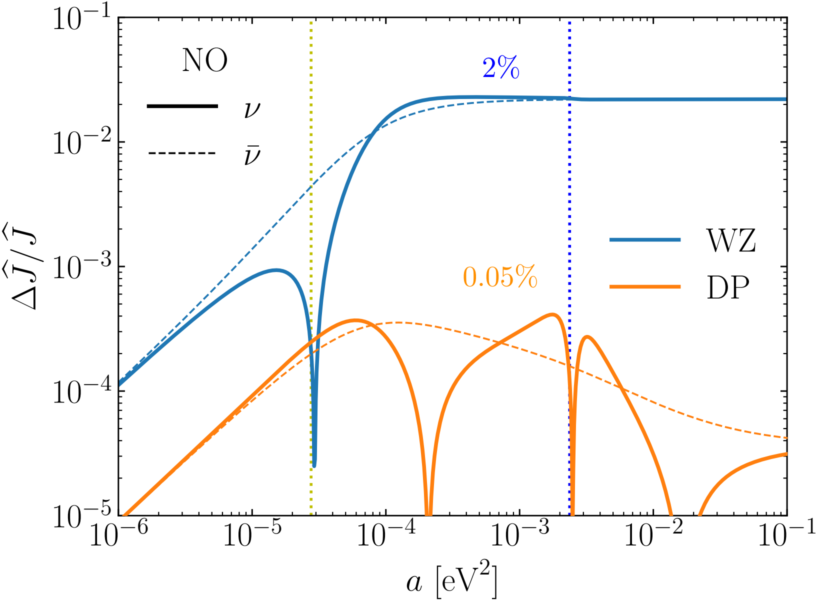
$$\mathcal{S}_{\rm atm} = \sqrt{(\cos 2\theta_{13} - a/\Delta m^2_{ee})^2 + \sin^2 2\theta_{13}}$$

$$\mathcal{S}_\odot = \sqrt{(\cos 2\theta_{12} - c^2_{13}a/\Delta m^2_{21})^2 + \sin^2 2\theta_{12}}$$

PBD, Parke, 1902.07185

See also X. Wang, S. Zhou, 1901.10882

CPV In Matter Approximation Precision



Is DMP the best?

Is DMP the best?

yes

Other Expressions

We were not the first to examine this problem.

- Madrid: drop $\frac{\Delta m^2_{21}}{\Delta m^2_{31}}$ and s_{13} terms; $\sim |\text{sum of two amplitudes}|^2$

$$P_{\mu e} = 4s^2_{23}s^2_{13}c^2_{13}\left(\frac{\Delta m^2_{31}}{b}\right)^2 \sin^2\Delta_b + 4c^2_{23}s^2_{12}c^2_{12}\left(\frac{\Delta m^2_{21}}{a}\right)^2 \sin^2\Delta_a + 8J_r\frac{\Delta m^2_{21}}{a}\frac{\Delta m^2_{31}}{b}\sin\Delta_a\sin\Delta_b\cos(\delta+\Delta_{31})\,, \quad b = a - \Delta m^2_{31}$$

A. Cervera, et al., hep-ph/0002108

Other Expressions

We were not the first to examine this problem.

- Madrid: drop $\frac{\Delta m^2_{21}}{\Delta m^2_{31}}$ and s_{13} terms; $\sim$ |sum of two amplitudes|^2

$$P_{\mu e} = 4s^2_{23}s^2_{13}c^2_{13}\left(\frac{\Delta m^2_{31}}{b}\right)^2 \sin^2\Delta_b + 4c^2_{23}s^2_{12}c^2_{12}\left(\frac{\Delta m^2_{21}}{a}\right)^2 \sin^2\Delta_a$$

$$+ 8J_r \frac{\Delta m^2_{21}}{a}\frac{\Delta m^2_{31}}{b}\sin\Delta_a \sin\Delta_b \cos(\delta + \Delta_{31}) \,, \quad b = a - \Delta m^2_{31}$$

A. Cervera, et al., hep-ph/0002108

E. Akhmedov, et al., hep-ph/0402175

A. Friedland, C. Lunardini, hep-ph/0606101

H. Nunokawa, S. Parke, J. Valle, 0710.0554

Other Expressions

We were not the first to examine this problem.

- Madrid: drop $\frac{\Delta m^2_{21}}{\Delta m^2_{31}}$ and s_{13} terms; $\sim |\text{sum of two amplitudes}|^2$

$$P_{\mu e} = 4s^2_{23}s^2_{13}c^2_{13}\left(\frac{\Delta m^2_{31}}{b}\right)^2 \sin^2\Delta_b + 4c^2_{23}s^2_{12}c^2_{12}\left(\frac{\Delta m^2_{21}}{a}\right)^2 \sin^2\Delta_a$$
$$+ 8J_r\frac{\Delta m^2_{21}}{a}\frac{\Delta m^2_{31}}{b}\sin\Delta_a\sin\Delta_b\cos(\delta+\Delta_{31})\ , \quad b = a - \Delta m^2_{31}$$

A. Cervera, et al., hep-ph/0002108

E. Akhmedov, et al., hep-ph/0402175

A. Friedland, C. Lunardini, hep-ph/0606101

H. Nunokawa, S. Parke, J. Valle, 0710.0554

- AKT: from mass basis rotated 12 then 23 converted into 13
  - Δm^2_{ee} appears all over the expressions

S. Agarwalla, Y. Kao, T. Takeuchi, 1302.6773

Other Expressions

- AM: Powers of $s_{13}^2 \simeq \frac{\Delta m_{21}^2}{\Delta m_{31}^2}$ through the 5/2 order

  K. Asano, H. Minakata, 1103.4387

- Various other expressions

  J. Arafune, M. Koike, J. Sato, hep-ph/9703351

  M. Freund, hep-ph/0103300

  E. Akhmedov, et al., hep-ph/0402175

Others...

Which is best?

Other Expressions

- AM: Powers of $s_{13}^2 \simeq \frac{\Delta m^2_{21}}{\Delta m^2_{31}}$ through the 5/2 order

  K. Asano, H. Minakata, 1103.4387

- Various other expressions

  J. Arafune, M. Koike, J. Sato, hep-ph/9703351

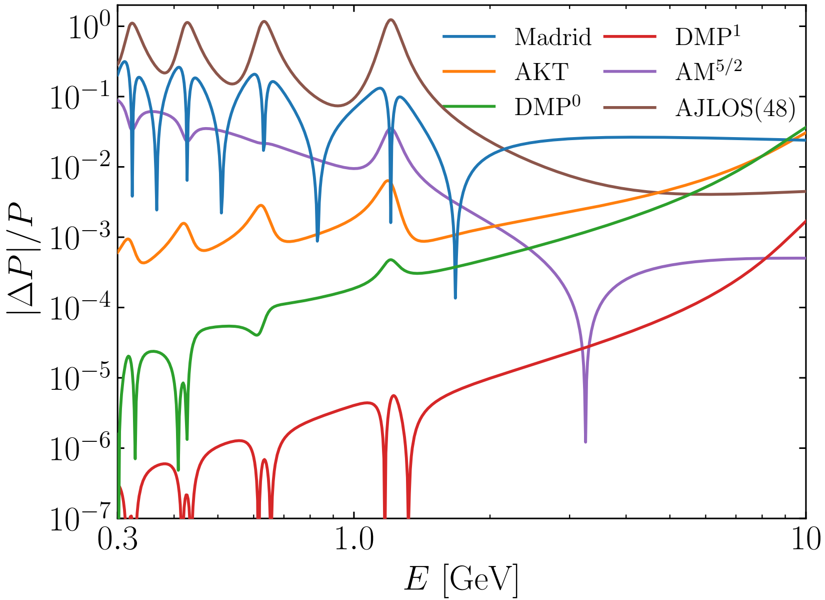
  M. Freund, hep-ph/0103300

  E. Akhmedov, et al., hep-ph/0402175

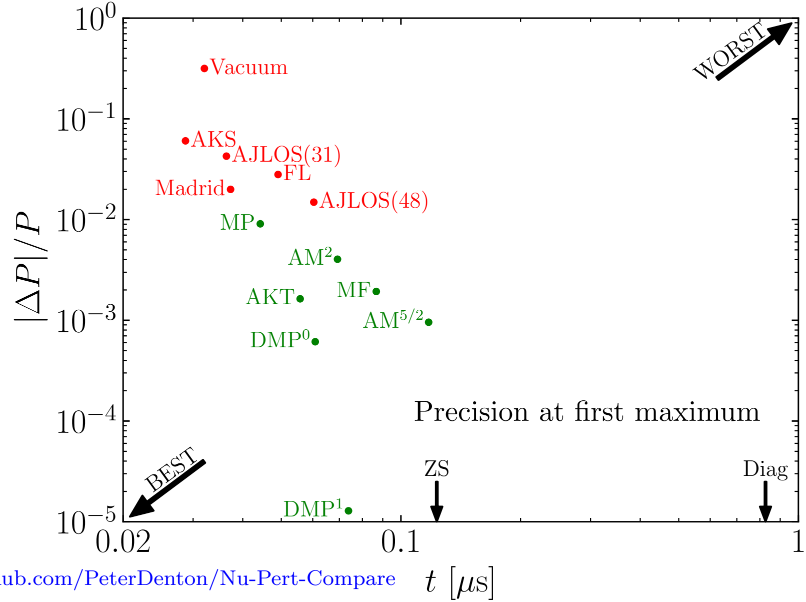
Others...

Which is best?
What does "best" mean?

Comparative Precision ($L = 1300$ km)



Speed $\propto$ Simplicity



github.com/PeterDenton/Nu-Pert-Compare

Proper Expansions

Parameter x is an *expansion parameter* iff

$$\lim_{x \to 0} P_{\text{approx}}(x) = P_{\text{exact}}(x = 0)$$

	ϵ	s_{13}	$a/\Delta m^2_{31}$	
Madrid(like)	✗	✗	✗	Cervera+, hep-ph/0002108
AKT	✓	✓	✓	Agarwalla+, 1302.6773
MP	✓	✗	✗	Minakata, Parke, 1505.01826
DMP	✓	✓	✓	PBD+, 1604.08167
AKS	✗	✗	✗	Arafune+, hep-ph/9703351
MF	✓	✗	✗	Freund, hep-ph/0103300
AJLOS(48)	✓	✗	✗	Akhmedov+, hep-ph/0402175
AM	✗	✗	✗	Asano, Minakata, 1103.4387

$$\epsilon \equiv \frac{\Delta m^2_{21}}{\Delta m^2_{ee}}$$

Comparative Review

- Many expressions in the literature (12 considered)
- Most are not at the 1% level
- Most are not exact in vacuum
- Changing the basis to remove level crossings seems best
  - AKT, (MP), DMP
  - Δm^2_{ee} naturally appears (regardless of the name)
- The order of rotations matters:
  - Constant 23 rotation, then in matter: 13, 12
- First order DMP corrections are quite simple

The Effective Δm^2_{ee} in Matter

$$\Delta m^2_{ee} = c^2_{12}\Delta m^2_{31} + s^2_{12}\Delta m^2_{32}$$

Δm^2_{ee} is an important quantity for understanding oscillations:

- Optimal expression for MBL reactor experiments

  H. Nunokawa, S. Parke, R. Zukanovich, hep-ph/0503283

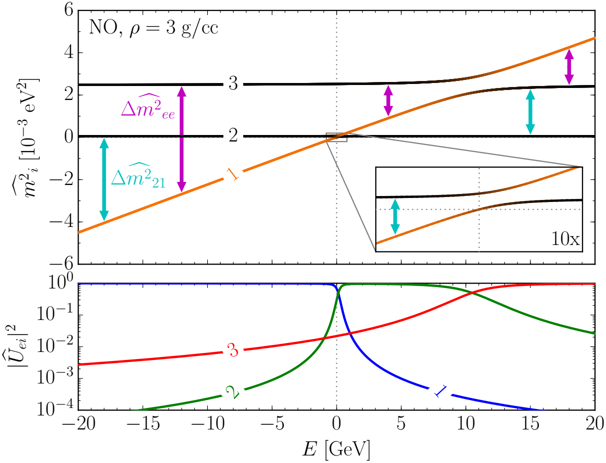
  S. Parke, 1601.07464

- Shows up naturally in DMP on long-baseline matter effect

How does Δm^2_{ee} evolve in matter?

Two-flavor approximation for ν_e disappearance in matter?

Asymptotic Evolution of $\Delta\widehat{m^2}_{ee}$



$$\Delta\widehat{m^2}_{ee} = \begin{cases} \widehat{m^2}_3 - \widehat{m^2}_1 & E \to -\infty \\ \widehat{m^2}_3 - \widehat{m^2}_2 & E \to +\infty \end{cases}$$

Intermediate Evolution of $\Delta\widehat{m^2}_{ee}$

$$\Delta\widehat{m^2}_{ee} = \begin{cases} \widehat{m^2}_3 - \widehat{m^2}_1 & E \to -\infty \\ \widehat{m^2}_3 - \widehat{m^2}_2 & E \to +\infty \end{cases}$$

Since $\widehat{m^2}_2(E \to -\infty) = \widehat{m^2}_1(E \to +\infty) =$ constant,
call $m_0^2 \equiv \Delta m_{21}^2 c_{12}^2$

Now we can define

$$\Delta\widehat{m^2}_{ee} \equiv \widehat{m^2}_3 - (\widehat{m^2}_1 + \widehat{m^2}_2 - m_0^2)$$

$$\Delta\widehat{m^2}_{ee} - \Delta m_{ee}^2 = (\widehat{m^2}_3 - m_3^2) - (\widehat{m^2}_1 - m_1^2) - (\widehat{m^2}_2 - m_2^2)$$

Easy to see that $\Delta\widehat{m^2}_{ee}(E = 0) = \Delta m_{ee}^2$

Relationship to vacuum expression?

Relationship to Vacuum Expression

In vacuum we can equivalently write:

$$\Delta m^2_{ee} = \begin{cases} c^2_{12}\Delta m^2_{31} + s^2_{12}\Delta m^2_{32} \\ m^2_3 - (m^2_1 + m^2_2 - m^2_0) \end{cases}$$

Elevate everything to matter equivalent,
except m^2_0 which we know we want to be a constant.

$$\Delta\widehat{m^2}_{ee} = \widehat{m^2}_3 - (\widehat{m^2}_1 + \widehat{m^2}_2 - m^2_0)$$

$$\Delta\widehat{m^2}_{EE} = c^2_{\widehat{12}}\Delta\widehat{m^2}_{31} + s^2_{\widehat{12}}\Delta\widehat{m^2}_{32}$$

The difference between these similar formulas:

$$\Delta_{Ee} = \widehat{m^2}_1 + c^2_{\widehat{12}}\Delta\widehat{m^2}_{21} - c^2_{12}\Delta m^2_{21}$$

$$< 0.3\%$$

A Third Option

Avoid $\cos(\frac{1}{3}\cos^{-1}\cdots)$, use DMP:

$$\widetilde{\Delta m^2}_{ee,\mathrm{DMP}} \equiv \Delta m^2_{ee}\sqrt{(\cos 2\theta_{13} - a/\Delta m^2_{ee})^2 + \sin^2 2\theta_{13}}$$

Which is best?

What does *best* mean?

Using one $\Delta m^2 \Rightarrow$ using a two-flavor picture:

$$P_{ee} \approx 1 - \sin^2 2\widehat{\theta}_{13} \sin^2 \frac{\Delta\widehat{m^2}_{ee} L}{4E}$$

$\Rightarrow$ want the first minimum correct
Take exact expression at $dP/dL = 0$ for a given E, then

$$\frac{\Delta\widehat{m^2}_{ee} L}{4E} = \frac{\pi}{2}$$

What does *best* mean?

Using one $\Delta m^2 \Rightarrow$ using a two-flavor picture:

$$P_{ee} \approx 1 - \sin^2 2\widehat{\theta}_{13} \sin^2 \frac{\Delta\widehat{m^2}_{ee} L}{4E}$$

$\Rightarrow$ want the first minimum correct

Take exact expression at $dP/dL = 0$ for a given E, then

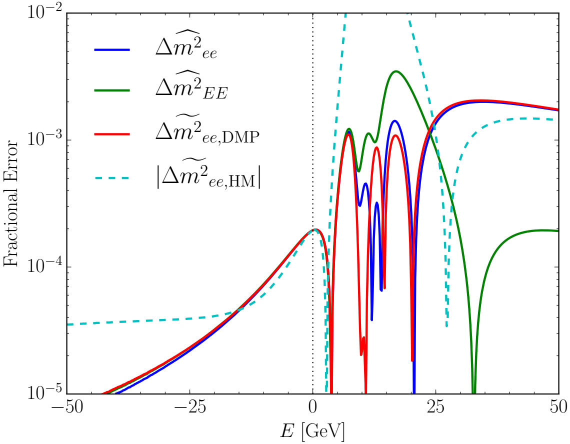
$$\frac{\Delta\widehat{m^2}_{ee} L}{4E} = \frac{\pi}{2}$$

Could do $dP/dE = 0$ for a given L,

H. Minakata, 1702.03332

but L/E show up together except where $a = a(E)$ appears,
and both $\widehat{\theta}_{13}$, $\Delta\widehat{m^2}_{ee}$ are complicated functions of a.

Comparison of Two-Flavor Precision



The $\widehat{\Delta m^2}_{ee}$ expression also leads to a simple rewriting of the eigenvalues.

HM: H. Minakata, 1702.03332
21 term in probability not included

$$\text{The winner is: } \widehat{\Delta m^2}_{ee} \equiv \widehat{m^2}_3 - (\widehat{m^2}_1 + \widehat{m^2}_2 - m_0^2)\ !$$

Precision is better than 0.06%

Depth of Oscillations

The depth of the minimum is well-described by

$$\sin^2 2\widehat{\theta_{13}} \approx \sin^2 2\theta_{13}\left(\frac{\Delta m^2_{ee}}{\widehat{\Delta m^2}_{ee}}\right)^2$$

$$\approx \frac{\sin^2 2\theta_{13}}{(\cos^2 2\theta_{13} - a/\Delta m^2_{ee})^2 + \sin^2 2\theta_{13}}$$

Using DMP

Depth of Oscillations

The depth of the minimum is well-described by

$$\sin^2 2\widehat{\theta}_{13} \approx \sin^2 2\theta_{13} \left(\frac{\Delta m^2_{ee}}{\widehat{\Delta m^2}_{ee}}\right)^2$$

$$\approx \frac{\sin^2 2\theta_{13}}{(\cos^2 2\theta_{13} - a/\Delta m^2_{ee})^2 + \sin^2 2\theta_{13}}$$

Using DMP

The disappearance probability in matter is well described by

$$P_{ee} \approx 1 - \sin^2 2\theta_{13} \left(\frac{\Delta m^2_{ee}}{\widehat{\Delta m^2}_{ee}}\right)^2 \sin^2 \frac{\widehat{\Delta m^2}_{ee} L}{4E}$$

$$\begin{aligned}\widehat{\Delta m^2}_{ee} \equiv{}& \widehat{m^2}_3 - (\widehat{m^2}_1 + \widehat{m^2}_2)\\ &- [m_3^2 - (m_1^2 + m_2^2)] + \Delta m^2_{ee}\end{aligned}$$

New Physics

DUNE and T2HK will unprecedented capabilities to test the three-neutrino oscillation picture

Extend DMP to new physics progress report:

- Sterile

S. Parke, X. Zhang, 1905.01356

- NSI?
- Neutrino decay?
- Decoherence?

Key Points

- Long-baseline oscillations are fundamentally three-flavor
- Include 1st order corrections in 0th order eigenvalues (Δm^2_{ee})
- Rotate **large terms first** $\Rightarrow$ PMNS order, removes level crossings
- 0th order probabilities: **same structure as vacuum** probabilities
- 0th order: **accurate** enough for current & future experiments
- DMP is the most precise while just as simple
- Exact and approximate CPV in matter are **simpler** than expected
- Same tools can also describe ν_e disappearance in matter: $\widehat{\Delta m^2}_{ee}$

Backups

Neutrino Oscillations in Vacuum: Disappearance

It is easy to calculate the *exact* disappearance expression in vacuum:

$$P(\nu_\alpha \to \nu_\alpha) = 1 - 4\sum_{i<j} |U_{\alpha i}|^2 |U_{\alpha j}|^2 \sin^2 \Delta_{ji}$$

For the electron case this expression is simple:

$$\begin{aligned} P(\nu_e \to \nu_e) = 1 & \\ & - 4c_{12}^2 s_{12}^2 c_{13}^4 \sin^2 \Delta_{21} \\ & - 4c_{12}^2 c_{13}^2 s_{13}^2 \sin^2 \Delta_{31} \\ & - 4s_{12}^2 c_{13}^2 s_{13}^2 \sin^2 \Delta_{32} \end{aligned}$$

$$\Delta_{ij} = \frac{\Delta m_{ij}^2 L}{4E}$$
$$\Delta m_{ij}^2 = m_i^2 - m_j^2$$

A Simple Solution

For two-flavor oscillations:

$$P(\nu_\alpha \to \nu_\beta) = \sin^2 2\theta \sin^2\left(\frac{\Delta m^2 L}{4E}\right)$$

- Solar: θ_{21}, Δm^2_{21}
- Reactor: θ_{13}, Δm^2_{ee}

Our Methodology

- Start with $\epsilon = \frac{\Delta m^2_{21}}{\Delta m^2_{ee}} = 0.03$
- Perform one fixed and two variable rotations: (θ_{23}, δ), $\widetilde{\theta}_{13}$, $\widetilde{\theta}_{12}$
- Write the probabilities with simple L/E dependence:

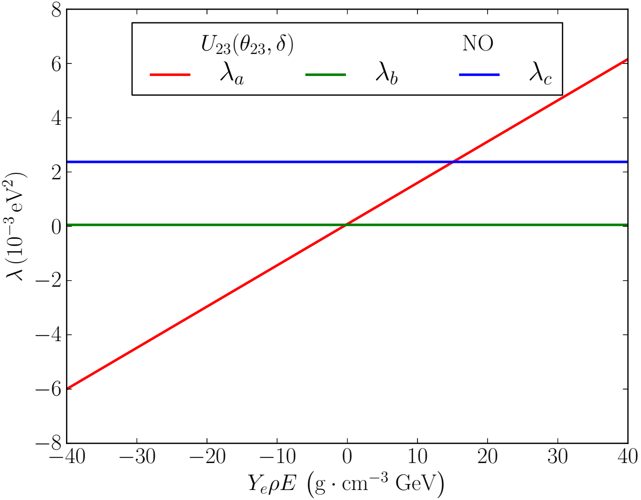
$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - \sum_{i<j} \Re\left[U_{\alpha i} U^*_{\beta i} U^*_{\alpha j} U_{\beta j}\right] \sin^2 \Delta_{ij}$$
$$+ 8\Im\left[U_{\alpha 1} U^*_{\beta 2} U^*_{\alpha 2} U_{\beta 1}\right] \sin \Delta_{32} \sin \Delta_{31} \sin \Delta_{21}$$

C. Jarlskog: PRL 55 (1985)

Nonvanishing Wronskian $\Rightarrow$ fewest number of L/E functions

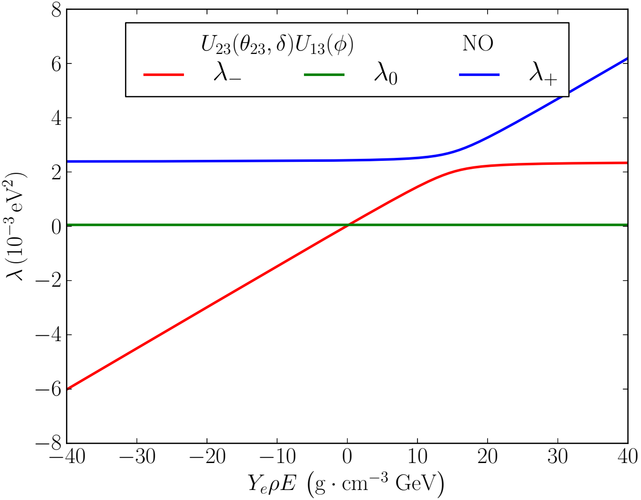
Clear that the CPV term is $\mathcal{O}[(L/E)^3]$ not $\mathcal{O}[(L/E)^1]$

Eigenvalues in Matter: Two Rotations are Needed



$$\widetilde{m^2}_a = a + (s_{13}^2 + \epsilon s_{12}^2)\Delta m_{ee}^2\,,\ \widetilde{m^2}_b = \epsilon c_{12}^2 \Delta m_{ee}^2\,,\ \widetilde{m^2}_c = (c_{13}^2 + \epsilon s_{12}^2)\Delta m_{ee}^2$$

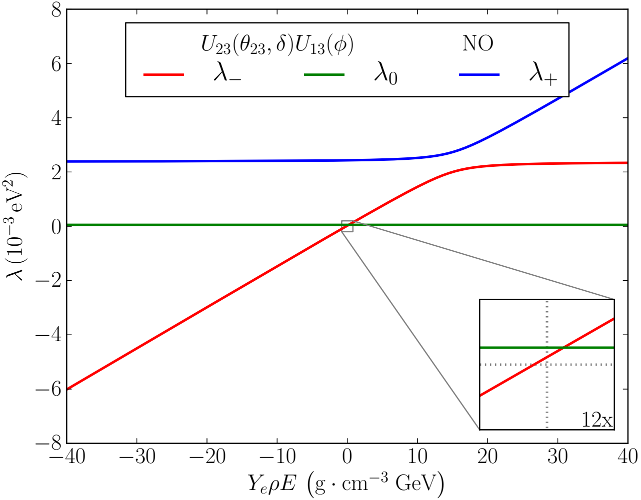
Eigenvalues in Matter: Two Rotations are Needed



$$\widetilde{m^2}_{\mp} = \frac{1}{2}\left[(\widetilde{m^2}_a + \widetilde{m^2}_c) \mp \mathrm{sgn}(\Delta m^2_{ee})\sqrt{(\widetilde{m^2}_c - \widetilde{m^2}_a)^2 + (2s_{13}c_{13}\Delta m^2_{ee})^2}\right]$$

$$\widetilde{m^2}_0 = \widetilde{m^2}_b$$

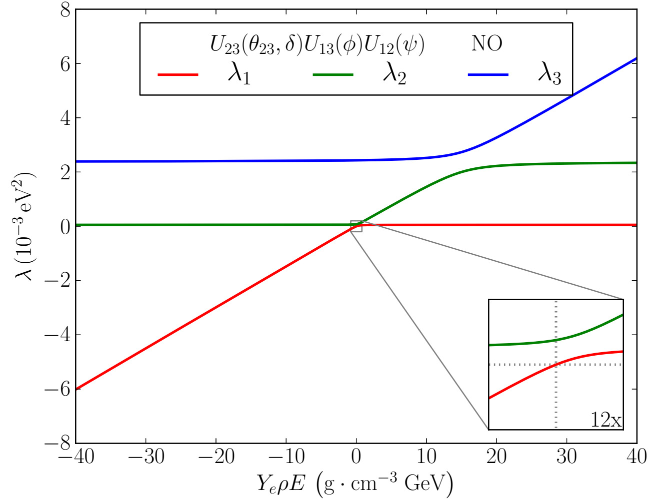
Eigenvalues in Matter: Two Rotations are Needed



$$\widetilde{m^2}_{\mp} = \frac{1}{2}\left[(\widetilde{m^2}_a + \widetilde{m^2}_c) \mp \mathrm{sgn}(\Delta m^2_{ee})\sqrt{(\widetilde{m^2}_c - \widetilde{m^2}_a)^2 + (2s_{13}c_{13}\Delta m^2_{ee})^2}\right]$$

$$\widetilde{m^2}_0 = \widetilde{m^2}_b$$

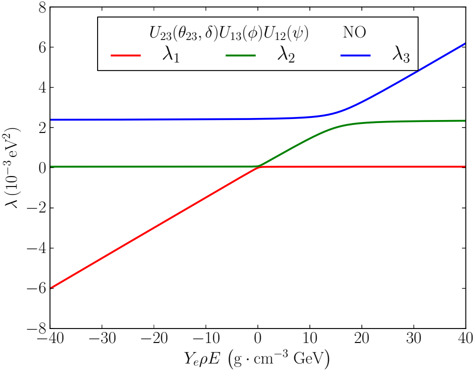
Eigenvalues in Matter: Two Rotations are Needed

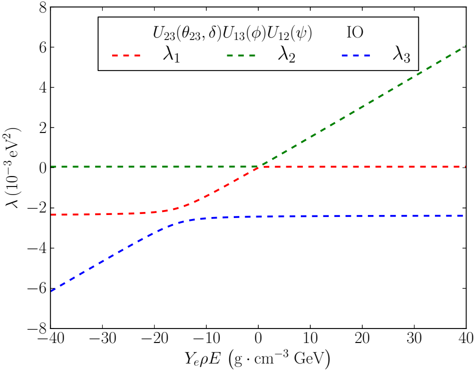


$$\widetilde{m^2}_{1,2}=\frac{1}{2}\left[(\widetilde{m^2}_0+\widetilde{m^2}_-)\mp\sqrt{(\widetilde{m^2}_0-\widetilde{m^2}_-)^2+(2\epsilon c_{(\widetilde{\theta}_{13}-\theta_{13})}c_{12}s_{12}\Delta m^2_{ee})^2}\right]$$

$$\widetilde{m^2}_3=\widetilde{m^2}_+$$

Eigenvalues in Matter: Mass Ordering





NO

$$\widetilde{m^2}_1 < \widetilde{m^2}_2 < \widetilde{m^2}_3$$

IO

$$\widetilde{m^2}_3 < \widetilde{m^2}_1 < \widetilde{m^2}_2$$

$1 + 2$ Rotations

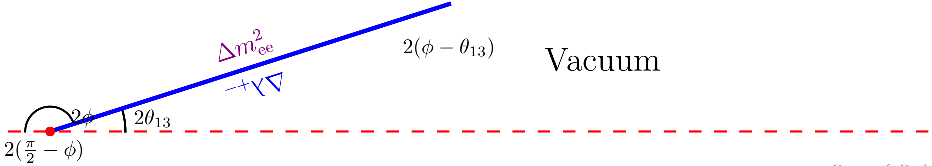
1. Perform a constant $U_{23}(\theta_{23}, \delta)$ rotation
   - U_{23} commutes with the matter potential
   - Resultant Hamiltonian is real
   - 'Expansion parameter' is $c_{13}s_{13} = 0.15$ at this point
2. Diagonalize the diagonal and $\mathcal{O}(\epsilon^0)$ off-diagonal terms with $U_{13}(\widetilde{\theta}_{13})$
   - $\widetilde{\theta}_{13}(a = 0) = \theta_{13}$
   - Expansion parameter is $c_{12}s_{12}\frac{\Delta m^2_{21}}{\Delta m^2_{ee}} = 0.015$

H. Minakata, S. Parke, 1505.01826

3. Diagonalize the terms non-zero in vacuum with $U_{12}(\widetilde{\theta}_{12})$
   - $\widetilde{\theta}_{12}(a = 0) = \theta_{12}$
   - Expansion parameter is now $\epsilon' = c_{12}s_{12}s_{(\widetilde{\theta}_{13}-\theta_{13})}\frac{\Delta m^2_{21}}{\Delta m^2_{ee}} < 0.015$
   - $\epsilon'(a = 0) = 0$

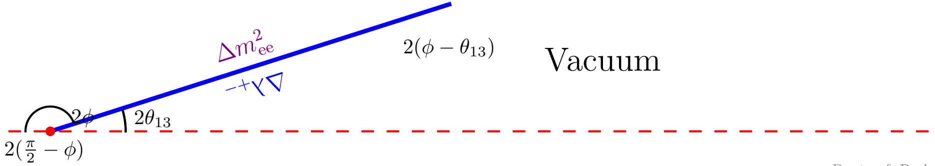
31 Triangle

$$\frac{a}{\Delta m^2_{\rm ee}\cos 2\theta_{13}} = 0.0$$



Denton & Parke

31 Triangle



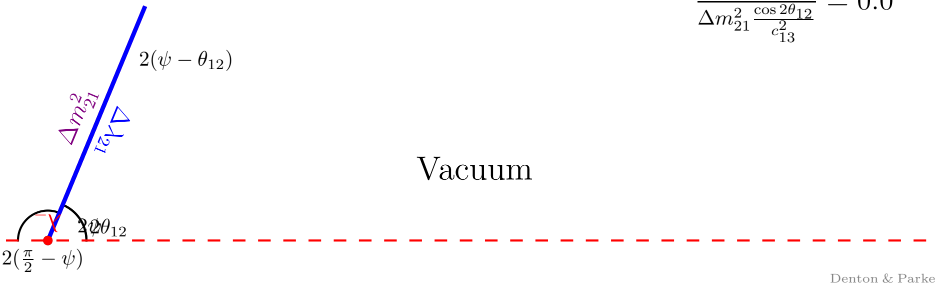
$$\frac{a}{\Delta m^2_{ee}\cos 2\theta_{13}} = 0.0$$

Vacuum

Denton & Parke

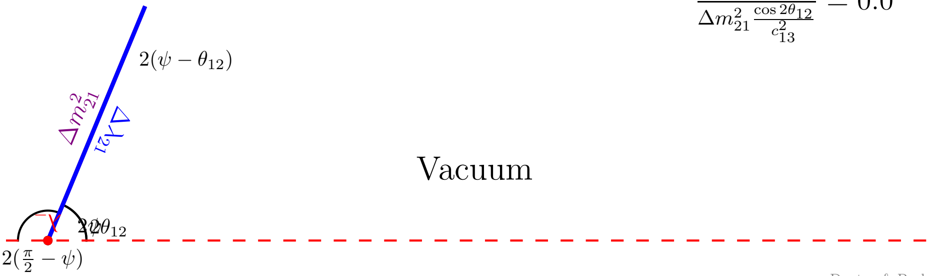
21 Triangle

$$\frac{a}{\Delta m^2_{21}\frac{\cos 2\theta_{12}}{c^2_{13}}} = 0.0$$



21 Triangle

$$\frac{a}{\Delta m^2_{21}\frac{\cos 2\theta_{12}}{c^2_{13}}} = 0.0$$



Denton & Parke

Exact Neutrino Oscillations in Matter: Mixing Angles

$$s^2_{\widehat{12}} = \frac{-\left[(\widehat{m^2}_2)^2 - \alpha\widehat{m^2}_2 + \beta\right]\Delta\widetilde{m^2}_{31}}{\left[(\widehat{m^2}_1)^2 - \alpha\widehat{m^2}_1 + \beta\right]\Delta\widetilde{m^2}_{32} - \left[(\widehat{m^2}_2)^2 - \alpha\widehat{m^2}_2 + \beta\right]\Delta\widetilde{m^2}_{31}}$$

$$s^2_{\widehat{13}} = \frac{(\widehat{m^2}_3)^2 - \alpha\widehat{m^2}_3 + \beta}{\Delta\widetilde{m^2}_{31}\Delta\widetilde{m^2}_{32}}$$

$$s^2_{\widehat{23}} = \frac{s^2_{23}E^2 + c^2_{23}F^2 + 2c_{23}s_{23}c_\delta EF}{E^2 + F^2}$$

$$e^{-i\widehat{\delta}} = \frac{c^2_{23}s^2_{23}\left(e^{-i\delta}E^2 - e^{i\delta}F^2\right) + \left(c^2_{23} - s^2_{23}\right)EF}{\sqrt{\left(s^2_{23}E^2 + c^2_{23}F^2 + 2EFc_{23}s_{23}c_\delta\right)\left(c^2_{23}E^2 + s^2_{23}F^2 - 2EFc_{23}s_{23}c_\delta\right)}}$$

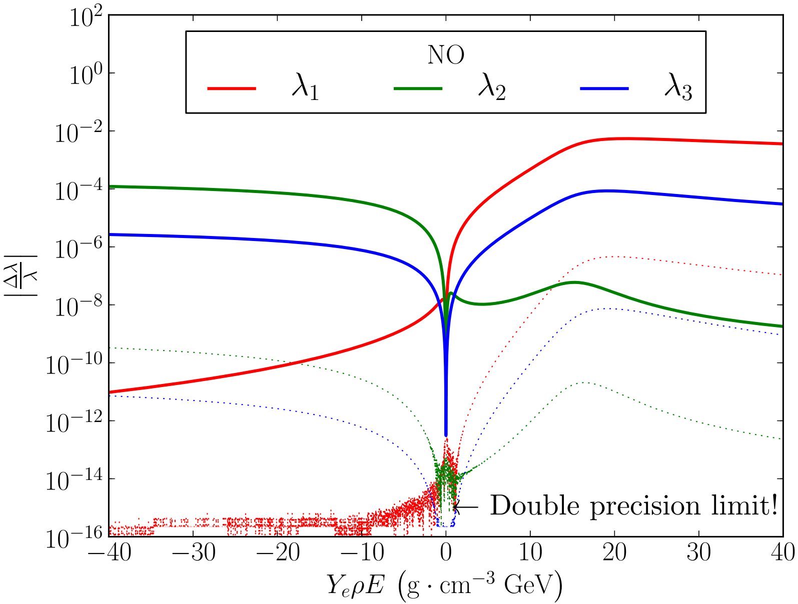
$$\alpha = c^2_{13}\Delta m^2_{31} + \left(c^2_{12}c^2_{13} + s^2_{13}\right)\Delta m^2_{21}\,, \;\; \beta = c^2_{12}c^2_{13}\Delta m^2_{21}\Delta m^2_{31}$$

$$E = c_{13}s_{13}\left[\left(\widehat{m^2}_3 - \Delta m^2_{21}\right)\Delta m^2_{31} - s^2_{12}\left(\widehat{m^2}_3 - \Delta m^2_{31}\right)\Delta m^2_{21}\right]$$

$$F = c_{12}s_{12}c_{13}\left(\widehat{m^2}_3 - \Delta m^2_{31}\right)\Delta m^2_{21}$$

H. Zaglauer, K. Schwarzer, Z.Phys. C40 (1988) 273

Eigenvalues: Precision



Hamiltonians

After a constant (θ_{23},δ) rotation, $2E\tilde{H}=$

$$\begin{pmatrix} \widetilde{m^2}_a & & s_{13}c_{13}\Delta m^2_{ee} \\ & \widetilde{m^2}_b & \\ s_{13}c_{13}\Delta m^2_{ee} & & \widetilde{m^2}_c \end{pmatrix} + \epsilon s_{12}c_{12}\Delta m^2_{ee} \begin{pmatrix} & c_{13} & \\ c_{13} & & -s_{13} \\ & -s_{13} & \end{pmatrix}$$

After a $U_{13}(\widetilde{\theta}_{13})$ rotation, $2E\hat{H}=$

$$\begin{pmatrix} \widetilde{m^2}_- & & \\ & \widetilde{m^2}_0 & \\ & & \widetilde{m^2}_+ \end{pmatrix} + \epsilon c_{12}s_{12}\Delta m^2_{ee} \begin{pmatrix} & c_{(\widetilde{\theta}_{13}-\theta_{13})} & \\ c_{(\widetilde{\theta}_{13}-\theta_{13})} & & s_{(\widetilde{\theta}_{13}-\theta_{13})} \\ & s_{(\widetilde{\theta}_{13}-\theta_{13})} & \end{pmatrix}$$

After a $U_{12}(\widetilde{\theta}_{12})$ rotation, $2E\check{H}=$

$$\begin{pmatrix} \widetilde{m^2}_1 & & \\ & \widetilde{m^2}_2 & \\ & & \widetilde{m^2}_3 \end{pmatrix} + \epsilon s_{(\widetilde{\theta}_{13}-\theta_{13})} s_{12}c_{12}\Delta m^2_{ee} \begin{pmatrix} & & -s_{\widetilde{12}} \\ & & c_{\widetilde{12}} \\ -s_{\widetilde{12}} & c_{\widetilde{12}} & \end{pmatrix}$$

Perturbative Expansion

Hamiltonian: $\check{H} = \check{H}_0 + \check{H}_1$

$$\check{H}_0 = \frac{1}{2E}\begin{pmatrix} \widetilde{m^2}_1 & & \\ & \widetilde{m^2}_2 & \\ & & \widetilde{m^2}_3 \end{pmatrix}, \quad \check{H}_1 = \epsilon' \frac{\Delta m^2_{ee}}{2E}\begin{pmatrix} & & -s_{\widetilde{12}} \\ & & c_{\widetilde{12}} \\ -s_{\widetilde{12}} & c_{\widetilde{12}} & \end{pmatrix}$$

Eigenvalues: $\widetilde{m^2}_i^{\mathrm{ex}} = \widetilde{m^2}_i + \widetilde{m^2}_i^{(1)} + \widetilde{m^2}_i^{(2)} + \ldots$

$$\widetilde{m^2}_i^{(1)} = 2E(\check{H}_1)_{ii} = 0$$

$$\widetilde{m^2}_i^{(2)} = \sum_{k\neq i} \frac{[2E(\check{H}_1)_{ik}]^2}{\Delta\widetilde{m^2}_{ik}}$$

Perturbative Expansion: Eigenvectors

Use vacuum expressions with $U \to V$ where

$$V = \widetilde{U} W$$

$\widetilde{U}$ is U with $\theta_{13} \to \widetilde{\theta}_{13}$ and $\theta_{12} \to \widetilde{\theta}_{12}$,

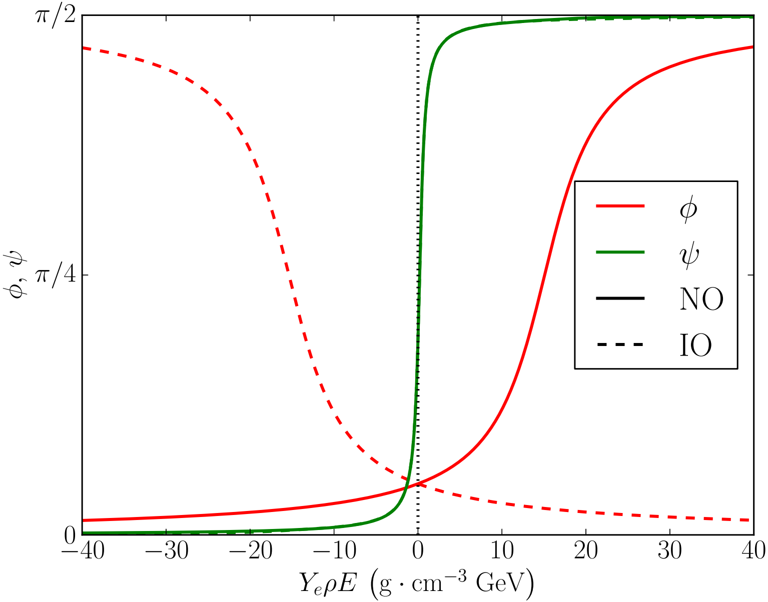
$W = W_0 + W_1 + W_2 + \ldots$

$$W_0 = \mathbb{1}$$

$$W_1 = \epsilon' \Delta m^2_{ee} \begin{pmatrix} & & -\frac{s_{\widetilde{12}}}{\widetilde{\Delta m^2}_{31}} \\ & & \frac{c_{\widetilde{12}}}{\widetilde{\Delta m^2}_{32}} \\ \frac{s_{\widetilde{12}}}{\widetilde{\Delta m^2}_{31}} & -\frac{c_{\widetilde{12}}}{\widetilde{\Delta m^2}_{32}} & \end{pmatrix}$$

$$W_2 = -\epsilon'^2 \frac{(\Delta m^2_{ee})^2}{2} \begin{pmatrix} \frac{s^2_{\widetilde{12}}}{(\widetilde{\Delta m^2}_{31})^2} & -\frac{s_{2\widetilde{12}}}{\widetilde{\Delta m^2}_{32} \widetilde{\Delta m^2}_{21}} & \\ \frac{s_{2\widetilde{12}}}{\widetilde{\Delta m^2}_{31} \widetilde{\Delta m^2}_{21}} & \frac{c^2_{\widetilde{12}}}{(\widetilde{\Delta m^2}_{32})^2} & \\ & & \left[\frac{c^2_{\widetilde{12}}}{(\widetilde{\Delta m^2}_{32})^2} + \frac{s^2_{\widetilde{12}}}{(\widetilde{\Delta m^2}_{31})^2}\right] \end{pmatrix}$$

The Two Matter Angles



Zeroth Order Coefficients

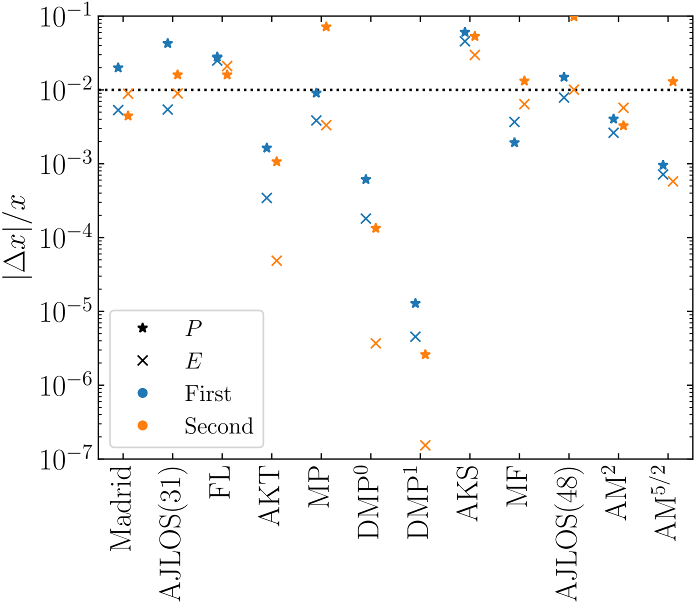
$$P_{\alpha\beta} = \delta_{\alpha\beta} + 4C_{21}^{\alpha\beta}\sin^2\Delta_{21} + 4C_{31}^{\alpha\beta}\sin^2\Delta_{31} + 4C_{32}^{\alpha\beta}\sin^2\Delta_{32} + 8D^{\alpha\beta}\sin\Delta_{21}\sin\Delta_{31}\sin\Delta_{32}$$

$\nu_\alpha \to \nu_\beta$	$(C_{21}^{\alpha\beta})^{(0)}$
$\nu_e \to \nu_e$	$-c_{\widetilde{13}}^4 s_{\widetilde{12}}^2 c_{\widetilde{12}}^2$
$\nu_\mu \to \nu_e$	$c_{\widetilde{13}}^2 s_{\widetilde{12}}^2 c_{\widetilde{12}}^2 (c_{23}^2 - s_{\widetilde{13}}^2 s_{23}^2) + c_{2\widetilde{12}} J_r^m c_\delta$
$\nu_\mu \to \nu_\mu$	$-(c_{23}^2 c_{\widetilde{12}}^2 + s_{23}^2 s_{\widetilde{13}}^2 s_{\widetilde{12}}^2)(c_{23}^2 s_{\widetilde{12}}^2 + s_{23}^2 s_{\widetilde{13}}^2 c_{\widetilde{12}}^2)$ $-2(c_{23}^2 - s_{\widetilde{13}}^2 s_{23}^2) c_{2\widetilde{12}} J_{rr}^m c_\delta + (2J_{rr}^m c_\delta)^2$

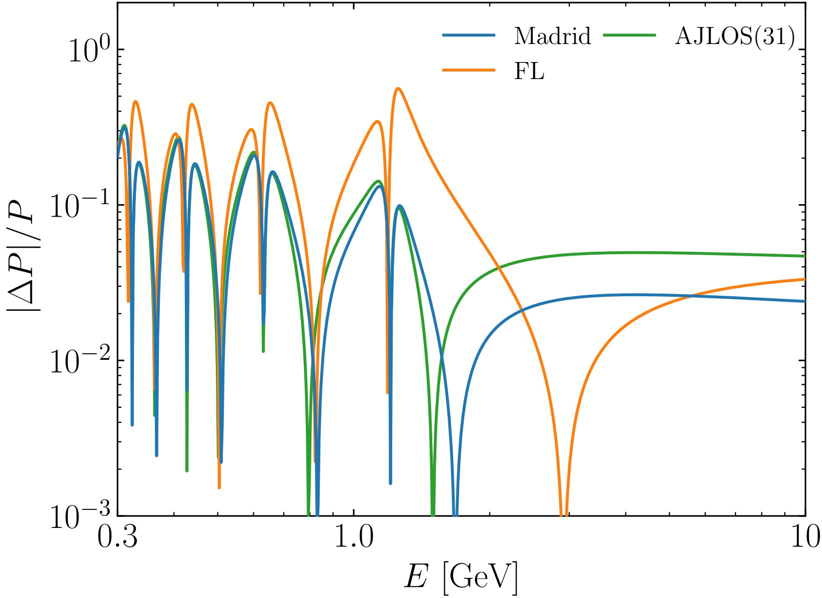
$\nu_\alpha \to \nu_\beta$	$(C_{31}^{\alpha\beta})^{(0)}$	$(D^{\alpha\beta})^{(0)}$
$\nu_e \to \nu_e$	$-c_{\widetilde{13}}^2 s_{\widetilde{13}}^2 c_{\widetilde{12}}^2$	0
$\nu_\mu \to \nu_e$	$s_{\widetilde{13}}^2 c_{\widetilde{13}}^2 c_{\widetilde{12}}^2 s_{23}^2 + J_r^m c_\delta$	$-J_r^m s_\delta$
$\nu_\mu \to \nu_\mu$	$-c_{\widetilde{13}}^2 s_{23}^2 (c_{23}^2 s_{\widetilde{12}}^2 + s_{23}^2 s_{\widetilde{13}}^2 c_{\widetilde{12}}^2)$ $-2s_{23}^2 J_r^m c_\delta$	0

$$J_r^m \equiv s_{\widetilde{12}} c_{\widetilde{12}} s_{\widetilde{13}} c_{\widetilde{13}}^2 s_{23} c_{23}, \quad J_{rr}^m \equiv J_r^m / c_{\widetilde{13}}^2$$

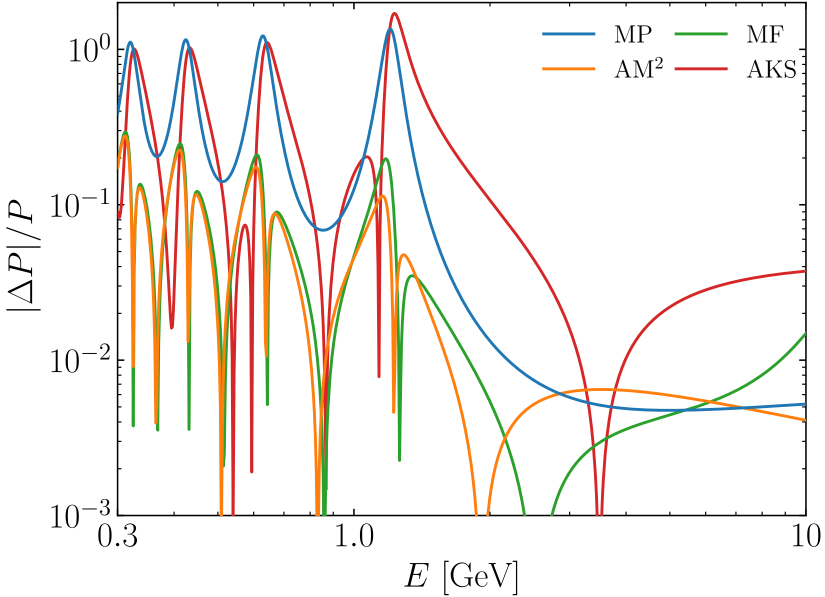
Comparative Precision: At the Peaks



Comparative Precision



Comparative Precision



Use DMP!

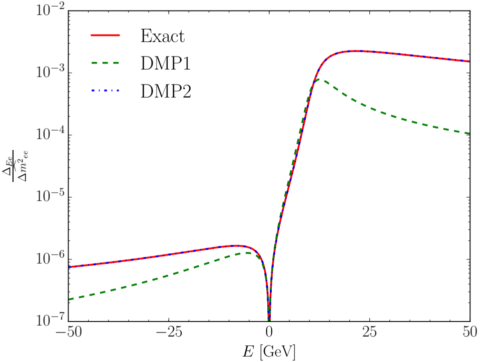
At zeroth order: $\Delta_{Ee}^{(0)} = 0$

At first order, only correction is to $\widetilde{\theta}_{12}$,

PBD, S. Parke, X. Zhang, 1806.01277

$$\Delta_{Ee}^{(1)} = t_{\widetilde{13}} s_{12}^2 c_{12}^2 \sin 2\theta_{13} a \frac{(\Delta m_{21}^2)^2}{\widetilde{\Delta m^2}_{32} \widetilde{\Delta m^2}_{31}}$$

At second order eigenvalues are also corrected, $\Delta_{Ee}^{(2)} = \cdots$



Error is quantified with DMP^2:

- First order isn't enough, ...
  - Second is
- Exact in vacuum

Angles in Matter

Angles receive corrections at first order:

$$\widetilde{\theta}_{12}^{(1)} = \epsilon' \Delta m_{ee}^2 s_{\widetilde{12}} c_{\widetilde{12}} \left(\frac{1}{\widetilde{\Delta m^2}_{32}} - \frac{1}{\widetilde{\Delta m^2}_{31}} \right)$$

$$\widetilde{\theta}_{13}^{(1)} = -\epsilon' \Delta m_{ee}^2 \frac{s_{\widetilde{13}}}{c_{\widetilde{13}}} \left(\frac{s_{\widetilde{12}}^2}{\widetilde{\Delta m^2}_{31}} + \frac{c_{\widetilde{12}}^2}{\widetilde{\Delta m^2}_{32}} \right)$$

$$\widetilde{\theta}_{23}^{(1)} = \epsilon' \Delta m_{ee}^2 \frac{c_\delta}{c_{\widetilde{13}}} \left(\frac{s_{\widetilde{12}}^2}{\widetilde{\Delta m^2}_{31}} + \frac{c_{\widetilde{12}}^2}{\widetilde{\Delta m^2}_{32}} \right)$$

$$\widetilde{\delta}^{(1)} = -\epsilon' \Delta m_{ee}^2 \frac{2 c_{2\widetilde{23}} s_\delta}{s_{2\widetilde{23}} c_{\widetilde{13}}} \left(\frac{s_{\widetilde{12}}^2}{\widetilde{\Delta m^2}_{31}} + \frac{c_{\widetilde{12}}^2}{\widetilde{\Delta m^2}_{32}} \right)$$

Second order: see paper