Abstract

We develop a perturbative framework for neutrino oscillations in uniform matter density so that the resulting oscillation probabilities are accurate for the complete matter potential versus baseline divided by neutrino energy plane. This extension also gives the *exact* oscillation probabilities in vacuum for all values of baseline divided by neutrino energy. The expansion parameter used is related to the ratio of the solar to the atmospheric Δm^2 scales but with a unique choice of the atmospheric Δm^2 such that certain first-order effects are taken into account in the zeroth-order Hamiltonian. Using a mixing matrix formulation, this framework has the exceptional feature that the neutrino oscillation probability in matter has the same structure as in vacuum, to all orders in the expansion parameter. It also contains all orders in the matter potential and $\sin\theta_{13}$. It facilitates immediate physical interpretation of the analytic results, and makes the expressions for the neutrino oscillation probabilities extremely compact and very accurate even at zeroth order in our perturbative expansion. The first and second order results are also given which improve the precision by approximately two or more orders of magnitude per perturbative order. We have also extended the framework by comparing the effect of additional rotations versus perturbations. Finally, we explore how Δm^2_{ee} is modified in matter.

Analytic and Compact Expressions for Neutrino Oscillations in Matter

Peter B. Denton

POND^2

December 4, 2018

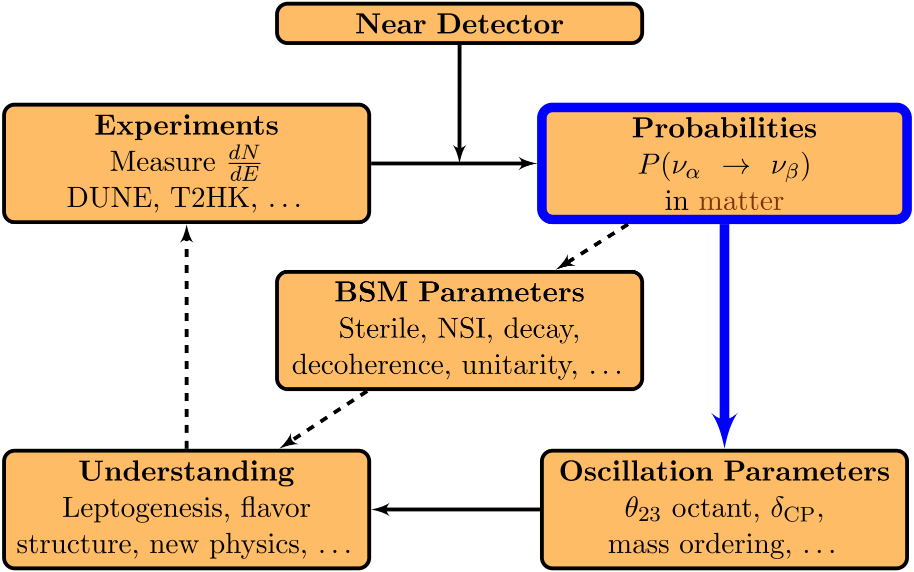
with S. Parke
and X. Zhang and H. Minakata.

1604.08167, 1806.01277, 1808.09453

github.com/PeterDenton/Nu-Pert

BROOKHAVEN
NATIONAL LABORATORY

A Theorist's Long-Baseline Picture



What is $P(\nu_\mu \to \nu_e)$?

$$P(\overset{(-)}{\nu}_\mu \to \overset{(-)}{\nu}_e) = |\mathcal{A}_{\mu e}|^2 \qquad \mathcal{A}_{\mu e} = \mathcal{A}_{31} + e^{\pm i\Delta_{32}}\mathcal{A}_{21}$$

$$\mathcal{A}_{31} = 2s_{13}c_{13}s_{23}\sin\Delta_{31}$$

$$\mathcal{A}_{21} = 2s_{12}c_{13}(c_{12}c_{23}e^{i\delta} - s_{12}s_{13}s_{23})\sin\Delta_{21}$$

$\Delta_{ij} = \Delta m^2{}_{ij}L/4E$

The Several Billion Dollar Question

What is $P(\nu_\mu \to \nu_e)$?

$$P(\overset{(-)}{\nu}_\mu \to \overset{(-)}{\nu}_e) = |\mathcal{A}_{\mu e}|^2 \qquad \mathcal{A}_{\mu e} = \mathcal{A}_{31} + e^{\pm i\Delta_{32}}\mathcal{A}_{21}$$

$$\mathcal{A}_{31} = 2s_{13}c_{13}s_{23}\sin\Delta_{31}$$

$$\mathcal{A}_{21} = 2s_{12}c_{13}(c_{12}c_{23}e^{i\delta} - s_{12}s_{13}s_{23})\sin\Delta_{21}$$

$\Delta_{ij} = \Delta m^2{}_{ij}L/4E$

...in matter?

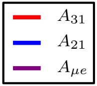
Now: NOvA, T2K, MINOS, ...
Upcoming: DUNE, T2HK, ...
Second maximum: T2HKK? ESSnuB? ...

Neutrino oscillations in vacuum: appearance

$\delta = 0.0\pi$

$\Delta_{32} = 0.5\pi$

NO



$P(\nu_\mu \to \nu_e) = A_{\mu e} A^*_{\mu e}$

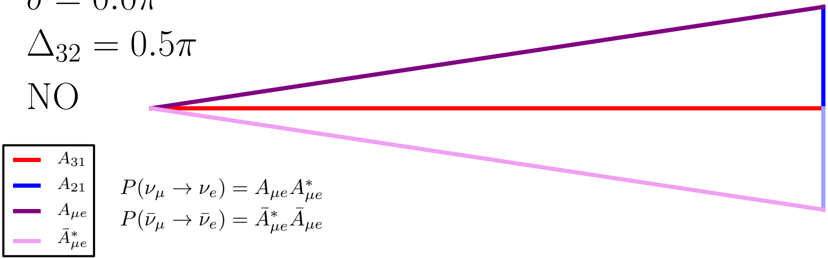
Denton & Parke

Neutrino oscillations in vacuum: appearance

$\delta = 0.0\pi$

$\Delta_{32} = 0.5\pi$

NO



$$P(\nu_\mu \to \nu_e) = A_{\mu e} A^*_{\mu e}$$
$$P(\bar{\nu}_\mu \to \bar{\nu}_e) = \bar{A}^*_{\mu e} \bar{A}_{\mu e}$$

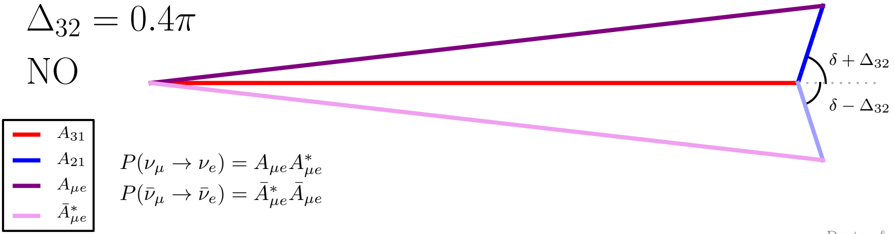
Denton & Parke

Neutrino oscillations in vacuum: appearance

$\delta = 0.0\pi$

$\Delta_{32} = 0.4\pi$

NO



$$P(\nu_\mu \to \nu_e) = A_{\mu e} A^*_{\mu e}$$
$$P(\bar{\nu}_\mu \to \bar{\nu}_e) = \bar{A}^*_{\mu e} \bar{A}_{\mu e}$$

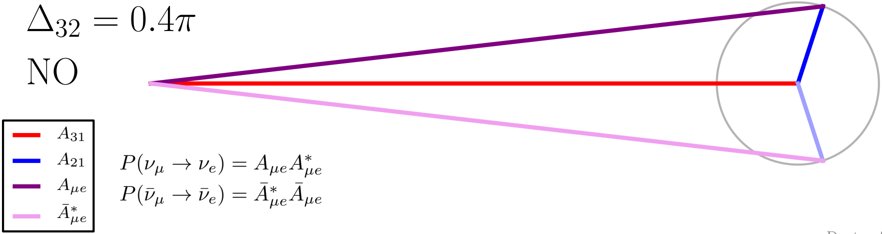
Denton & Parke

Neutrino oscillations in vacuum: appearance

$\delta = 0.0\pi$

$\Delta_{32} = 0.4\pi$

NO



$$P(\nu_\mu \to \nu_e) = A_{\mu e} A^*_{\mu e}$$
$$P(\bar{\nu}_\mu \to \bar{\nu}_e) = \bar{A}^*_{\mu e} \bar{A}_{\mu e}$$

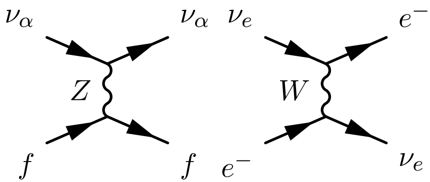
Denton & Parke

Matter Effects Matter

Call Schrödinger equation's eigenvalues m_i^2 and eigenvectors U_i.

$$\mathcal{A}(\nu_\alpha \to \nu_\beta) = \sum_i U_{\alpha i}^* U_{\beta i} e^{-i m_i^2 L/2E} \qquad\qquad P = |\mathcal{A}|^2$$

In matter ν's propagate in a new basis that depends on $a \propto \rho E$.



L. Wolfenstein, PRD 17 (1978)

Eigenvalues: $m_i^2 \to \widetilde{m^2}_i(a)$

Eigenvectors are given by $\theta_{ij} \to \widetilde{\theta}_{ij}(a)$ $\qquad \Leftarrow \qquad$ Unitarity

Hamiltonian Dynamics

$$H=\frac{1}{2E}\left[U\begin{pmatrix}0&&\\&\Delta m^2_{21}&\\&&\Delta m^2_{31}\end{pmatrix}U^\dagger+\begin{pmatrix}a&&\\&0&\\&&0\end{pmatrix}\right]$$

$a=2\sqrt{2}G_FN_eE$

Find eigenvalues and eigenvectors:

$$H=\frac{1}{2E}\widetilde{U}\begin{pmatrix}0&&\\&\Delta\widetilde{m^2}_{21}&\\&&\Delta\widetilde{m^2}_{31}\end{pmatrix}\widetilde{U}^\dagger$$

Computationally works, but we can do better than a $\boxed{\textbf{black box}}$...

Analytic expression?

Eigenvalues Analytically: The Exact Solution

Solve the cubic characteristic equation.

$$\widetilde{m^2}_1 = \frac{A}{3} - \frac{1}{3}\sqrt{A^2 - 3B}S - \frac{\sqrt{3}}{3}\sqrt{A^2 - 3B}\sqrt{1 - S^2}$$

$$\widetilde{m^2}_2 = \frac{A}{3} - \frac{1}{3}\sqrt{A^2 - 3B}S + \frac{\sqrt{3}}{3}\sqrt{A^2 - 3B}\sqrt{1 - S^2}$$

$$\widetilde{m^2}_3 = \frac{A}{3} + \frac{2}{3}\sqrt{A^2 - 3B}S$$

$$A = \Delta m^2_{21} + \Delta m^2_{31} + a$$

$$B = \Delta m^2_{21}\Delta m^2_{31} + a\left[c^2_{13}\Delta m^2_{31} + (c^2_{12}c^2_{13} + s^2_{13})\Delta m^2_{21}\right]$$

$$C = a\Delta m^2_{21}\Delta m^2_{31}c^2_{12}c^2_{13}$$

$$S = \cos\left\{\frac{1}{3}\cos^{-1}\left[\frac{2A^3 - 9AB + 27C}{2\left(A^2 - 3B\right)^{3/2}}\right]\right\}$$

H. Zaglauer, K. Schwarzer, Z. Phys. C Particles and Fields, 40 (1988)

Traded one $\boxed{\textbf{black box}}$ for another...

Alternative Solutions

Perturbative expansion:

- Small matter potential: $a/\Delta m^2$

Y. Li, Y. Wang, Z-z. Xing, 1605.00900

- s_{13}, s_{13}^2

A. Cervera, et al., hep-ph/0002108

H. Minakata, 0910.5545

K. Asano, H. Minakata, 1103.4387

- $\Delta m_{21}^2/\Delta m_{31}^2 \sim 0.03$

J. Arafune, J. Sato, hep-ph/9607437

A. Cervera, et al., hep-ph/0002108

M. Freund, hep-ph/0103300

E. Akhmedov, et al., hep-ph/0402175

M. Blennow, A. Smirnov, 1306.2903

H. Minakata, S. Parke, 1505.01826

PBD, H. Minakata, S. Parke, 1604.08167

Change of basis:

PBD, S. Parke, X. Zhang, 1806.01277

A Tale of Two Tools

Split the Hamiltonian into:

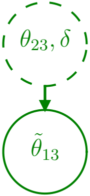
- Large, diagonal part (H_0)
- Small, off-diagonal part (H_1)

- Improves precision at zeroth order
- Naturally leads to using $\Delta m^2_{ee} \equiv c^2_{12}\Delta m^2_{31} + s^2_{12}\Delta m^2_{32}$

H. Nunokawa, S. Parke, R. Zukanovich, hep-ph/0503283

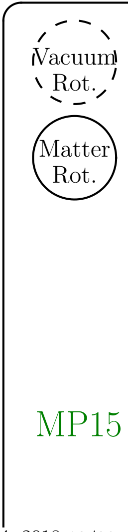
1. **Rotations**:
   - A two-flavor rotation only requires solving a quadratic
   - Diagonalize away the big off-diagonal terms
   - Follows the order of the PMNS matrix
2. **Perturbative expansion**:
   - Smallness parameter is $|\epsilon'| \leq 0.015$
   - Correct eigenvalues ($\widetilde{m^2}_i$) and eigenvectors ($\widetilde{\theta_{ij}}$)
   - Eigenvalues already include 1$^{\text{st}}$ order corrections at 0$^{\text{th}}$ order
   - Can improve the precision to arbitrary order

Atmospheric Resonance

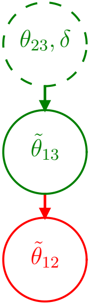


1. $U_{23}(\theta_{23}, \delta)$ commutes with matter potential
2. Largest off-diagonal term:
   $s_{13}c_{13}\Delta m^2_{ee}$ in the 1-3 position

- Eigenvalues still cross at the solar resonance:
  - No perturbation theory there
- Smallness parameter:
  - After U_{23}: $s_{13}c_{13} = 0.15$
  - After U_{13}: $s_{12}c_{12}\frac{\Delta m^2_{21}}{\Delta m^2_{ee}} = 0.015$

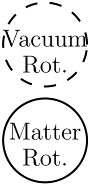


Solar Resonance



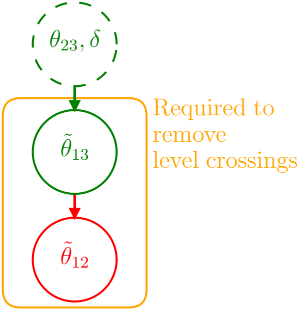
3. Largest off-diagonal term:
   $s_{12}c_{12}c_{\widetilde{\theta}_{13}-\theta_{13}}\Delta m^2_{21}$ in the 1-2 position
   - Largest except for ν's above the atmospheric resonance

- $|\epsilon'| < 0.015$, zero in vacuum
- Perturbation theory valid everywhere now
- Rotation order matches PMNS
- Take vacuum expressions, replace θ_{13}, θ_{12}, and Δm^2_{ij}
- Extremely precise $|\Delta P/P| < 10^{-3}$



MP15
DMP16

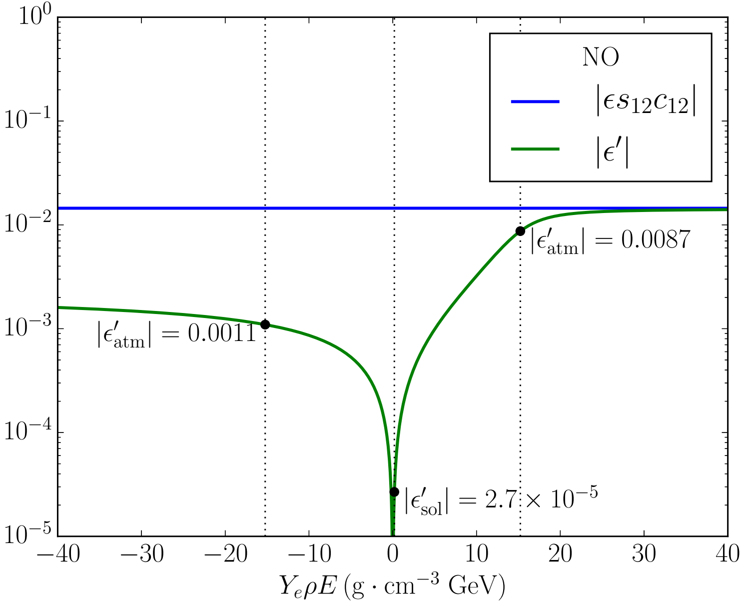
Solar Resonance



3. Largest off-diagonal term:
   $s_{12}c_{12}c_{\widetilde{\theta}_{13}-\theta_{13}}\Delta m^2_{21}$ in the 1-2 position
   - Largest except for ν's above the atmospheric resonance

- $|\epsilon'| < 0.015$, zero in vacuum
- Perturbation theory valid everywhere now
- Rotation order matches PMNS
- Take vacuum expressions, replace θ_{13}, θ_{12}, and Δm^2_{ij}
- Extremely precise $|\Delta P/P| < 10^{-3}$

Expansion Parameter



Probability in Matter: DMP 0^{th}

$$\text{Vacuum} \quad \Rightarrow \quad \widetilde{\text{Matter}}$$

$$P_{\alpha\beta}(\Delta m^2_{21}, \Delta m^2_{31}, \theta_{13}, \theta_{12}, \theta_{23}, \delta) \Rightarrow P_{\alpha\beta}(\Delta \widetilde{m^2}_{21}, \Delta \widetilde{m^2}_{31}, \widetilde{\theta}_{13}, \widetilde{\theta}_{12}, \theta_{23}, \delta)$$

Same expression, 4 new variables.

Probability in Matter: DMP 0^{th}

$$\text{Vacuum} \qquad \Rightarrow \qquad \widetilde{\text{Matter}}$$

$$P_{\alpha\beta}(\Delta m^2_{21}, \Delta m^2_{31}, \theta_{13}, \theta_{12}, \theta_{23}, \delta) \Rightarrow P_{\alpha\beta}(\Delta\widetilde{m^2}_{21}, \Delta\widetilde{m^2}_{31}, \widetilde{\theta}_{13}, \widetilde{\theta}_{12}, \theta_{23}, \delta)$$

Same expression, 4 new variables.

$$\cos 2\widetilde{\theta}_{13} = \frac{\Delta m^2_{ee}\cos 2\theta_{13} - a}{\Delta\widetilde{m^2}_{ee}}$$

$$\Delta\widetilde{m^2}_{ee} = \Delta m^2_{ee}\sqrt{(\cos 2\theta_{13} - a/\Delta m^2_{ee})^2 + \sin^2 2\theta_{13}}$$

Probability in Matter: DMP 0^{th}

$$\text{Vacuum} \quad \Rightarrow \quad \widetilde{\text{Matter}}$$

$$P_{\alpha\beta}(\Delta m^2_{21}, \Delta m^2_{31}, \theta_{13}, \theta_{12}, \theta_{23}, \delta) \Rightarrow P_{\alpha\beta}(\Delta\widetilde{m^2}_{21}, \Delta\widetilde{m^2}_{31}, \widetilde{\theta}_{13}, \widetilde{\theta}_{12}, \theta_{23}, \delta)$$

Same expression, 4 new variables.

$$\cos 2\widetilde{\theta}_{13} = \frac{\Delta m^2_{ee}\cos 2\theta_{13} - a}{\Delta\widetilde{m^2}_{ee}}$$

$$\Delta\widetilde{m^2}_{ee} = \Delta m^2_{ee}\sqrt{(\cos 2\theta_{13} - a/\Delta m^2_{ee})^2 + \sin^2 2\theta_{13}}$$

$$\cos 2\widetilde{\theta}_{12} = \frac{\Delta m^2_{21}\cos 2\theta_{12} - a_{12}}{\Delta\widetilde{m^2}_{21}}\,, \qquad a_{12} = (a + \Delta m^2_{ee} - \Delta\widetilde{m^2}_{ee})/2$$

$$\Delta\widetilde{m^2}_{21} = \Delta m^2_{21}\sqrt{(\cos 2\theta_{12} - a_{12}/\Delta m^2_{21})^2 + \cos^2(\widetilde{\theta}_{13} - \theta_{13})\sin^2 2\theta_{12}}$$

Probability in Matter: DMP 0^{th}

Vacuum $\Rightarrow$ $\widetilde{\text{Matter}}$

$$P_{\alpha\beta}(\Delta m^2_{21}, \Delta m^2_{31}, \theta_{13}, \theta_{12}, \theta_{23}, \delta) \Rightarrow P_{\alpha\beta}(\Delta\widetilde{m^2}_{21}, \Delta\widetilde{m^2}_{31}, \widetilde\theta_{13}, \widetilde\theta_{12}, \theta_{23}, \delta)$$

Same expression, 4 new variables.

$$\cos 2\widetilde\theta_{13} = \frac{\Delta m^2_{ee}\cos 2\theta_{13} - a}{\Delta\widetilde{m^2}_{ee}}$$

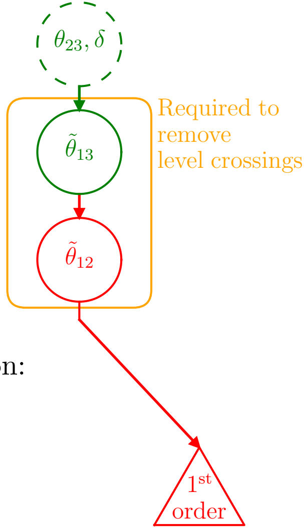
$$\Delta\widetilde{m^2}_{ee} = \Delta m^2_{ee}\sqrt{(\cos 2\theta_{13} - a/\Delta m^2_{ee})^2 + \sin^2 2\theta_{13}}$$

$$\cos 2\widetilde\theta_{12} = \frac{\Delta m^2_{21}\cos 2\theta_{12} - a_{12}}{\Delta\widetilde{m^2}_{21}}, \qquad a_{12} = (a + \Delta m^2_{ee} - \Delta\widetilde{m^2}_{ee})/2$$

$$\Delta\widetilde{m^2}_{21} = \Delta m^2_{21}\sqrt{(\cos 2\theta_{12} - a_{12}/\Delta m^2_{21})^2 + \cos^2(\widetilde\theta_{13} - \theta_{13})\sin^2 2\theta_{12}}$$

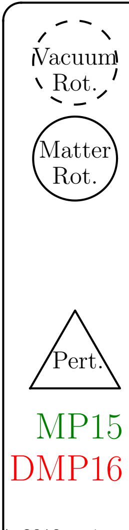
$$\Delta\widetilde{m^2}_{31} = \Delta m^2_{31} + \frac{1}{4}a + \frac{1}{2}(\Delta\widetilde{m^2}_{21} - \Delta m^2_{21}) + \frac{3}{4}(\Delta\widetilde{m^2}_{ee} - \Delta m^2_{ee})$$

Improve with Perturbation



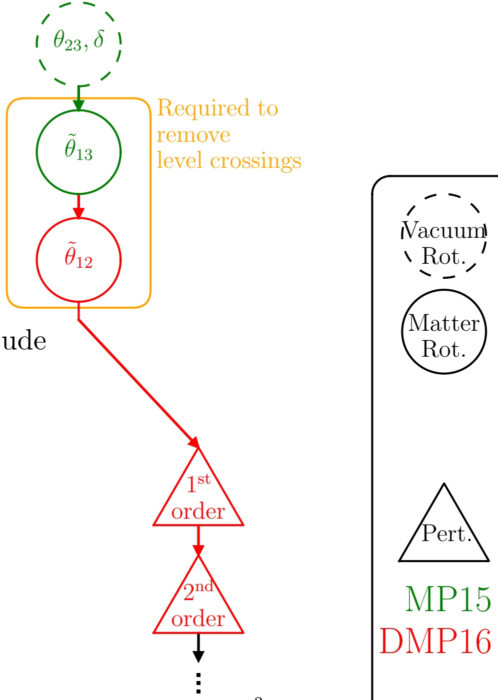
4. $\gtrsim 2$ orders of magnitude of improvement in precision: $|\Delta P/P| < 10^{-6}$

- Eigenvalues need no correction
- Compact form utilizes a $\widetilde{m^2}_1 \leftrightarrow \widetilde{m^2}_2$, $\widetilde{\theta}_{12} \leftrightarrow \widetilde{\theta}_{12} \pm \pi/2$ symmetry

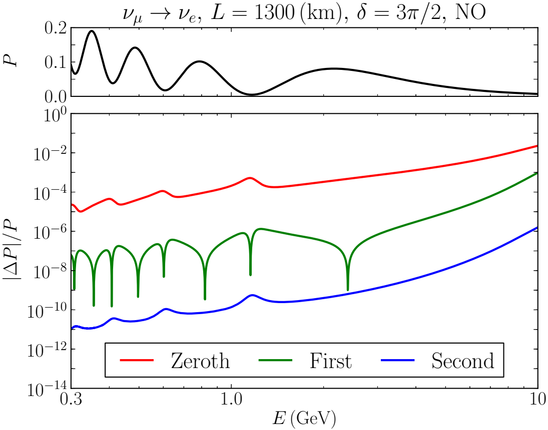


Higher Orders

5. $\gtrsim 2$ more orders of magnitude of improvement per order: $|\Delta P/P| < 10^{-9}$, ...

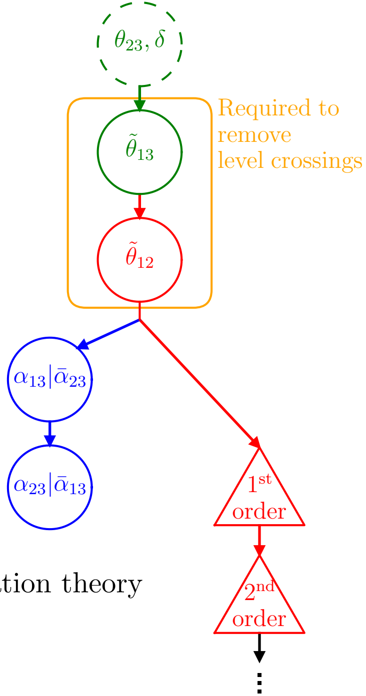


Precision



DUNE: NO, $\delta = 3\pi/2$		First min	First max
$P(\nu_\mu \to \nu_e)$		0.0047	0.081
E (GeV)		1.2	2.2
$\frac{\lvert\Delta P\rvert}{P}$	Zeroth	5×10^{-4}	4×10^{-4}
	First	3×10^{-7}	2×10^{-7}
	Second	6×10^{-10}	5×10^{-10}

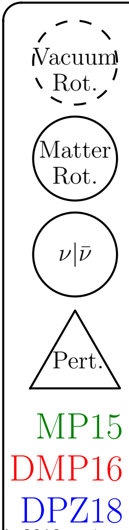
More Rotations



Instead continue to diagonalize large terms

4. 1-3 sector for ν's
   2-3 sector for $\bar\nu$'s
5. Then opposite

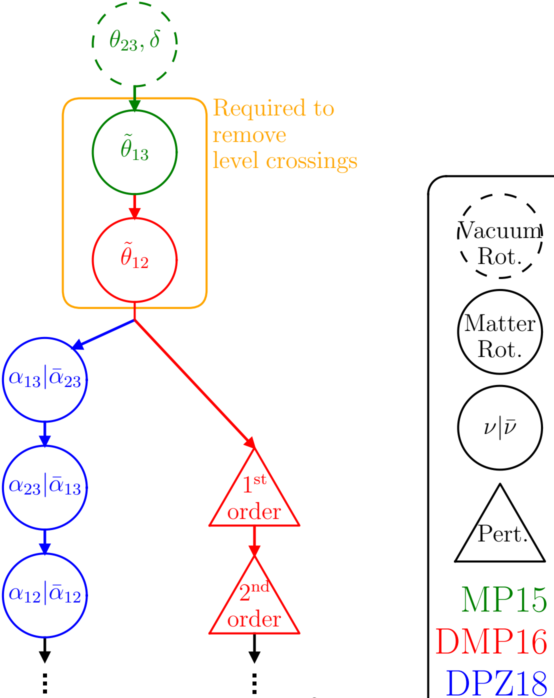
- 2 additional rotations
  $\equiv$ 1 order of perturbation theory



Even More Rotations

6. 1-2 sector for either $\nu/\bar{\nu}$

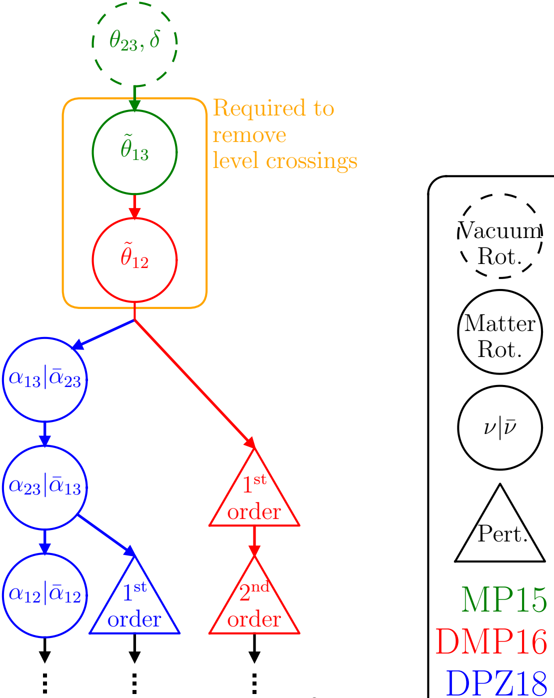
- 3 additional rotations $\equiv$ 2 orders of pert. th.



More Options

6. Perturbation theory after 2 additional rotations

- 2 additional rotations
+ 1 order of pert. th.
≡ 2 orders of pert. th.



The Effective Δm^2_{ee} in Matter

$$\Delta m^2_{ee} = c^2_{12}\Delta m^2_{31} + s^2_{12}\Delta m^2_{32}$$

Δm^2_{ee} is an important quantity for understanding oscillations:

- Optimal expression for reactor experiments

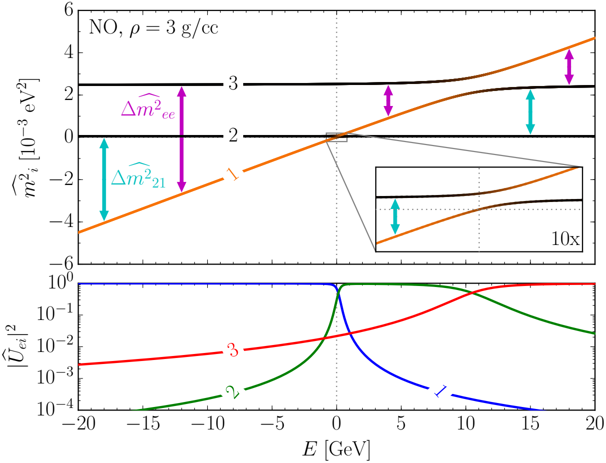
  H. Nunokawa, S. Parke, R. Zukanovich, hep-ph/0503283

  S. Parke, 1601.07464

- Shows up naturally in DMP on long-baseline matter effect

How does Δm^2_{ee} evolve in matter?

Asymptotic Evolution of $\Delta\widehat{m^2}_{ee}$



$$\Delta\widehat{m^2}_{ee} = \begin{cases} \widehat{m^2}_3 - \widehat{m^2}_1 & E \to -\infty \\ \widehat{m^2}_3 - \widehat{m^2}_2 & E \to +\infty \end{cases}$$

Intermediate Evolution of $\Delta\widehat{m^2}_{ee}$

$$\Delta\widehat{m^2}_{ee} = \begin{cases} \widehat{m^2}_3 - \widehat{m^2}_1 & E \to -\infty \\ \widehat{m^2}_3 - \widehat{m^2}_2 & E \to +\infty \end{cases}$$

Since $\widehat{m^2}_2(E \to -\infty) = \widehat{m^2}_1(E \to +\infty) = \text{constant}$,
call $m_0^2 \equiv \Delta m_{21}^2 c_{12}^2$

Now we can define

$$\Delta\widehat{m^2}_{ee} \equiv \widehat{m^2}_3 - (\widehat{m^2}_1 + \widehat{m^2}_2 - m_0^2)$$

$$\Delta\widehat{m^2}_{ee} - \Delta m_{ee}^2 = (\widehat{m^2}_3 - m_3^2) - (\widehat{m^2}_1 - m_1^2) - (\widehat{m^2}_2 - m_2^2)$$

Easy to see that $\Delta\widehat{m^2}_{ee}(E = 0) = \Delta m_{ee}^2$

Relationship to vacuum expression?

Relationship to Vacuum Expression

In vacuum we can equivalently write:

$$\Delta m^2_{ee} = \begin{cases} c^2_{12}\Delta m^2_{31} + s^2_{12}\Delta m^2_{32} \\ m^2_3 - (m^2_1 + m^2_2 - m^2_0) \end{cases}$$

Elevate everything to matter equivalent,
except m^2_0 which we know we want to be a constant.

$$\Delta\widehat{m^2}_{ee} = \widehat{m^2}_3 - (\widehat{m^2}_1 + \widehat{m^2}_2 - m^2_0)$$

$$\Delta\widehat{m^2}_{EE} = c^2_{\widehat{12}}\Delta\widehat{m^2}_{31} + s^2_{\widehat{12}}\Delta\widehat{m^2}_{32}$$

The difference between these similar formulas:

$$\Delta_{Ee} = \widehat{m^2}_1 + c^2_{\widehat{12}}\Delta\widehat{m^2}_{21} - c^2_{12}\Delta m^2_{21}$$

Use DMP!

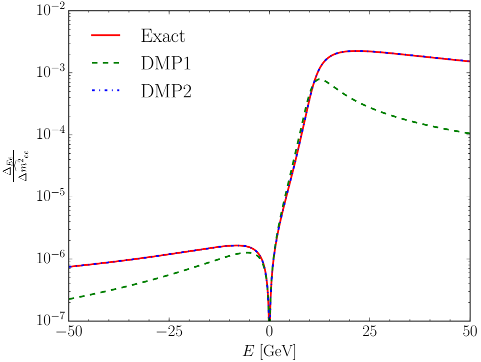
At zeroth order: $\Delta_{Ee}^{(0)} = 0$

At first order, only correction is to $\widetilde{\theta}_{12}$,

PBD, S. Parke, X. Zhang, 1806.01277

$$\Delta_{Ee}^{(1)} = t_{\widetilde{13}} s_{12}^2 c_{12}^2 \sin 2\theta_{13} a \frac{(\Delta m_{21}^2)^2}{\widetilde{\Delta m^2}_{32} \widetilde{\Delta m^2}_{31}}$$

At second order eigenvalues are also corrected, $\Delta_{Ee}^{(2)} = \cdots$

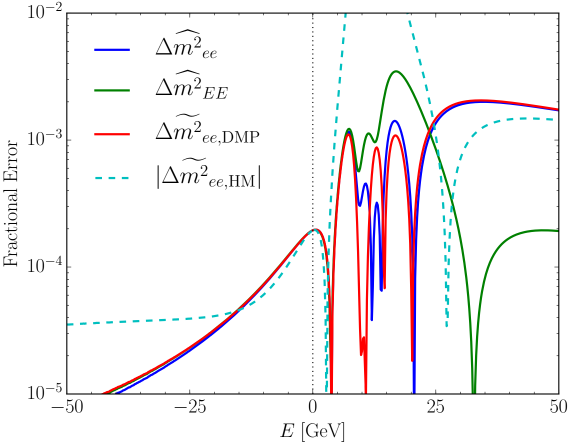


Avoid $\cos(\frac{1}{3}\cos^{-1} X)$, use DMP:

$$\widetilde{\Delta m^2}_{ee,\mathrm{DMP}} / \Delta m_{ee}^2 \equiv \sqrt{(\cos 2\theta_{13} - a/\Delta m_{ee}^2)^2 + \sin^2 2\theta_{13}}$$

Which is best?

Comparison of Two-Flavor Precision



The $\widehat{\Delta m^2}_{ee}$ expression also leads to a simple rewriting of the eigenvalues.

HM: H. Minakata, 1702.03332
21 term in probability not included

The winner is: $\widehat{\Delta m^2}_{ee} \equiv \widehat{m^2}_3 - (\widehat{m^2}_1 + \widehat{m^2}_2 - m_0^2)$!

Precision is better than 0.06%

Key Points

- Include 1^{st} order corrections in 0^{th} order eigenvalues (Δm^2_{ee})
- Rotate **large terms first** $\Rightarrow$ PMNS order, removes level crossings
- All channels, energies, and baselines handled simultaneously
- 0^{th} order probabilities: **same structure as vacuum** probabilities
- 0^{th} order: **accurate** enough for current & future experiments
- **Further precision** through perturbation and/or more rotations
- Elevate Δm^2_{ee} to include the matter effect, DMP makes it simple
- New physics? Xining Zhang on steriles next!

Backups

Neutrino Oscillations in Vacuum: Disappearance

It is easy to calculate the *exact* disappearance expression in vacuum:

$$P(\nu_\alpha \to \nu_\alpha) = 1 - 4 \sum_{i<j} |U_{\alpha i}|^2 |U_{\alpha j}|^2 \sin^2 \Delta_{ji}$$

For the electron case this expression is simple:

$$\begin{aligned} P(\nu_e \to \nu_e) = 1 & \\ & - 4c_{12}^2 s_{12}^2 c_{13}^4 \sin^2 \Delta_{21} \\ & - 4c_{12}^2 c_{13}^2 s_{13}^2 \sin^2 \Delta_{31} \\ & - 4s_{12}^2 c_{13}^2 s_{13}^2 \sin^2 \Delta_{32} \end{aligned}$$

$$\Delta_{ij} = \frac{\Delta m_{ij}^2 L}{4E}$$
$$\Delta m_{ij}^2 = m_i^2 - m_j^2$$

A Simple Solution

For two-flavor oscillations:

$$P(\nu_\alpha \to \nu_\beta) = \sin^2 2\theta \sin^2\left(\frac{\Delta m^2 L}{4E}\right)$$

- Solar: θ_{21}, Δm^2_{21}
- Reactor: θ_{13}, Δm^2_{ee}

Neutrino Oscillation Parameters Status

Six parameters:

1. $\theta_{13} = 8.6^\circ$
2. $\theta_{12} = 34^\circ$
3. $\Delta m^2_{21} = 7.4 \times 10^{-5}$ eV^2
4. $\theta_{23} \sim 45^\circ$ (octant)
5. $|\Delta m^2_{31}| = 2.5 \times 10^{-3}$ eV^2 (mass ordering)
6. $\delta = ???$

NuFIT, 1811.05487

PMNS order allows for easy measurement of θ_{13} and θ_{12}.

θ_{23} and δ_{CP} require full three-flavor description.

Alternative Solutions: Example

$$P_0=\sin^2\theta_{23}\frac{\sin^2 2\theta_{13}}{\hat{C}^2}\sin^2(\hat{\Delta}\hat{C}), \tag{36a}$$

$$P_{\sin\delta}=\frac{1}{2}\alpha\frac{\sin\delta\cos\theta_{13}\sin 2\theta_{12}\sin 2\theta_{13}\sin 2\theta_{23}}{\hat{A}\hat{C}\cos\theta_{13}^2}\sin(\hat{C}\hat{\Delta})$$

$$\times\{\cos(\hat{C}\hat{\Delta})-\cos((1+\hat{A})\hat{\Delta})\}, \tag{36b}$$

$$P_{\cos\delta}=\frac{1}{2}\alpha\frac{\cos\delta\cos\theta_{13}\sin 2\theta_{12}\sin 2\theta_{13}\sin 2\theta_{23}}{\hat{A}\hat{C}\cos\theta_{13}^2}\sin(\hat{C}\hat{\Delta})$$

$$\times\{\sin((1+\hat{A})\hat{\Delta})\mp\sin(\hat{C}\hat{\Delta})\}, \tag{36c}$$

$$P_1=-\alpha\frac{1-\hat{A}\cos 2\theta_{13}}{\hat{C}^3}\sin^2\theta_{12}\sin^2 2\theta_{13}\sin^2\theta_{23}\hat{\Delta}$$

$$\times\sin(2\hat{\Delta}\hat{C})+\alpha\frac{2\hat{A}(-\hat{A}+\cos 2\theta_{13})}{\hat{C}^4}$$

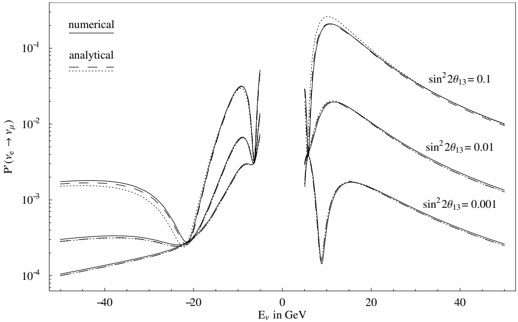
$$\times\sin^2\theta_{12}\sin^2 2\theta_{13}\sin^2\theta_{23}\sin^2(\hat{\Delta}\hat{C}), \tag{36d}$$

$$P_2=\alpha\frac{\mp 1+\hat{C}\pm\hat{A}\cos 2\theta_{13}}{2\hat{C}^2\hat{A}\cos^2\theta_{13}}\cos\theta_{13}\sin 2\theta_{12}\sin 2\theta_{13}$$

$$\times\sin 2\theta_{23}\sin^2(\hat{\Delta}\hat{C}), \tag{36e}$$

$$P_3=\alpha^2\frac{2\hat{C}\cos^2\theta_{23}\sin^2 2\theta_{12}}{\hat{A}^2\cos^2\theta_{13}(\mp\hat{A}+\hat{C}\pm\cos 2\theta_{13})}$$

$$\times\sin^2\left(\frac{1}{2}(1+\hat{A}\mp\hat{C})\hat{\Delta}\right). \tag{36f}$$



M. Freund, hep-ph/0103300

Our Methodology

- Start with $\epsilon = \frac{\Delta m^2_{21}}{\Delta m^2_{ee}} = 0.03$
- Perform one fixed and two variable rotations: (θ_{23}, δ), $\widetilde{\theta}_{13}$, $\widetilde{\theta}_{12}$
- Write the probabilities with simple L/E dependence:

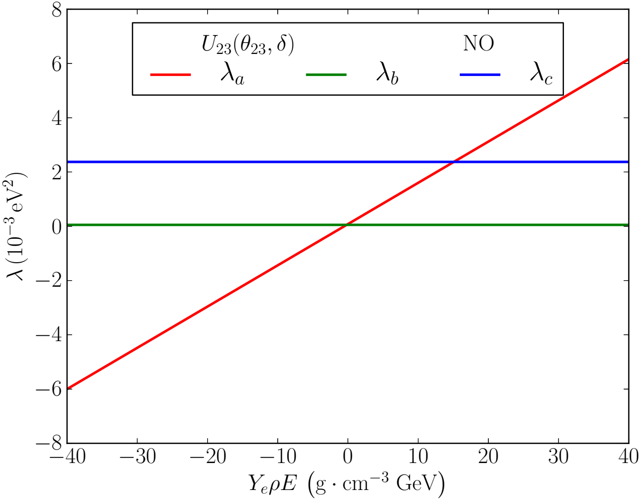
$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - \sum_{i<j} \Re\left[U_{\alpha i} U^*_{\beta i} U^*_{\alpha j} U_{\beta j}\right] \sin^2 \Delta_{ij}$$
$$+ 8\Im\left[U_{\alpha 1} U^*_{\beta 2} U^*_{\alpha 2} U_{\beta 1}\right] \sin \Delta_{32} \sin \Delta_{31} \sin \Delta_{21}$$

C. Jarlskog: PRL 55 (1985)

Nonvanishing Wronskian $\Rightarrow$ fewest number of L/E functions

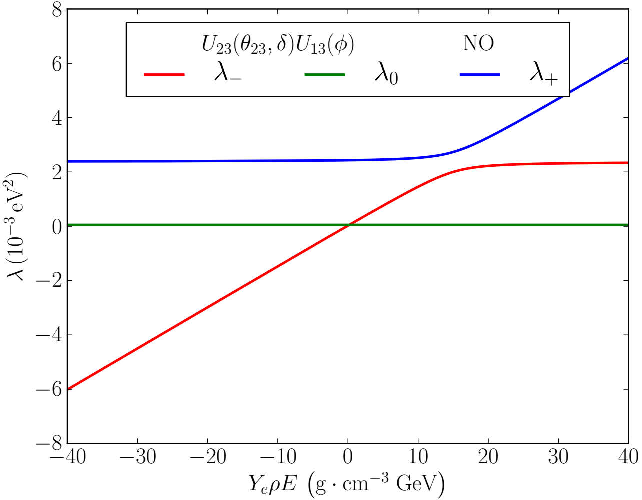
Clear that the CPV term is $\mathcal{O}[(L/E)^3]$ not $\mathcal{O}[(L/E)^1]$

Eigenvalues in Matter: Two Rotations are Needed



$$\widetilde{m^2}_a = a + (s_{13}^2 + \epsilon s_{12}^2)\Delta m_{ee}^2\,,\ \widetilde{m^2}_b = \epsilon c_{12}^2 \Delta m_{ee}^2\,,\ \widetilde{m^2}_c = (c_{13}^2 + \epsilon s_{12}^2)\Delta m_{ee}^2$$

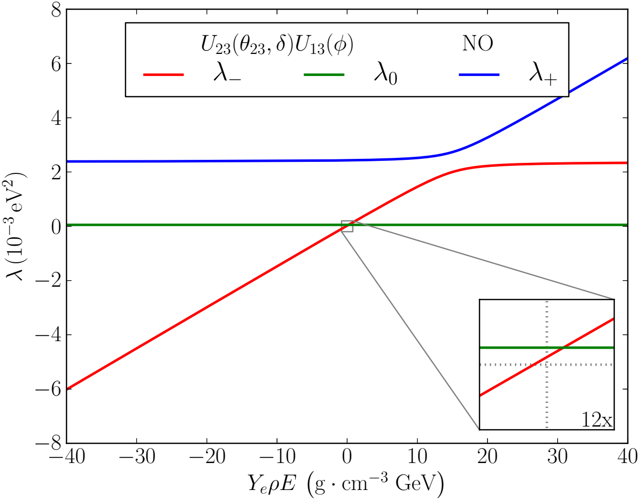
Eigenvalues in Matter: Two Rotations are Needed



$$\widetilde{m^2}_{\mp} = \frac{1}{2}\left[(\widetilde{m^2}_a + \widetilde{m^2}_c) \mp \mathrm{sgn}(\Delta m^2_{ee})\sqrt{(\widetilde{m^2}_c - \widetilde{m^2}_a)^2 + (2s_{13}c_{13}\Delta m^2_{ee})^2}\right]$$

$$\widetilde{m^2}_0 = \widetilde{m^2}_b$$

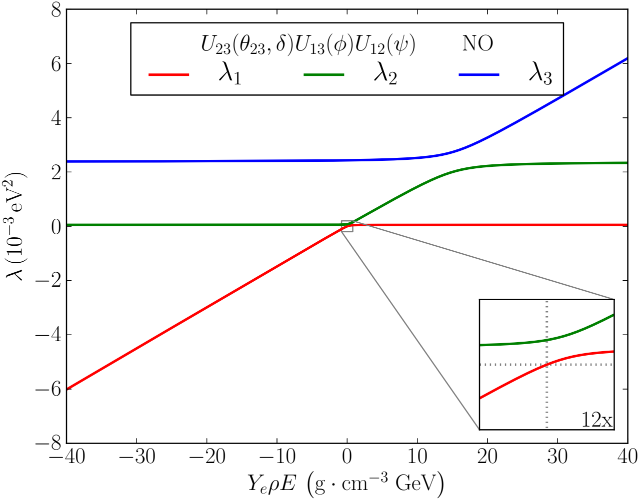
Eigenvalues in Matter: Two Rotations are Needed



$$\widetilde{m^2}_{\mp} = \frac{1}{2}\left[(\widetilde{m^2}_a + \widetilde{m^2}_c) \mp \mathrm{sgn}(\Delta m^2_{ee})\sqrt{(\widetilde{m^2}_c - \widetilde{m^2}_a)^2 + (2s_{13}c_{13}\Delta m^2_{ee})^2}\right]$$

$$\widetilde{m^2}_0 = \widetilde{m^2}_b$$

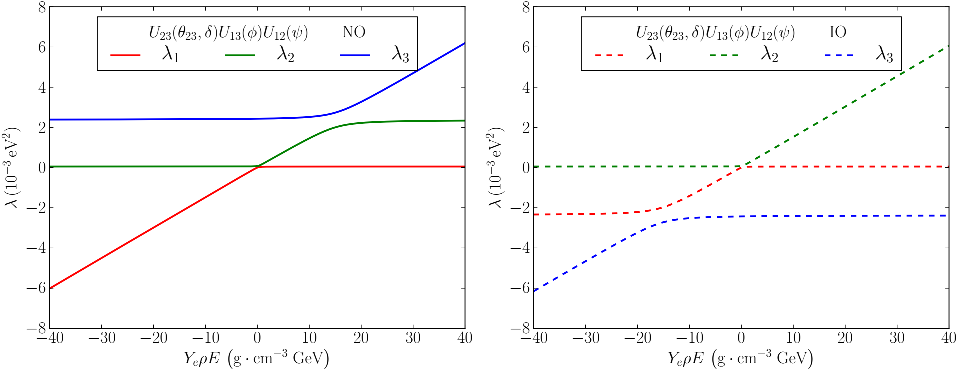
Eigenvalues in Matter: Two Rotations are Needed



$$\widetilde{m^2}_{1,2}=\frac{1}{2}\left[(\widetilde{m^2}_0+\widetilde{m^2}_-)\mp\sqrt{(\widetilde{m^2}_0-\widetilde{m^2}_-)^2+(2\epsilon c_{(\widetilde{\theta}_{13}-\theta_{13})}c_{12}s_{12}\Delta m^2_{ee})^2}\right]$$

$$\widetilde{m^2}_3=\widetilde{m^2}_+$$

Eigenvalues in Matter: Mass Ordering



NO

$$\widetilde{m^2}_1 < \widetilde{m^2}_2 < \widetilde{m^2}_3$$

IO

$$\widetilde{m^2}_3 < \widetilde{m^2}_1 < \widetilde{m^2}_2$$

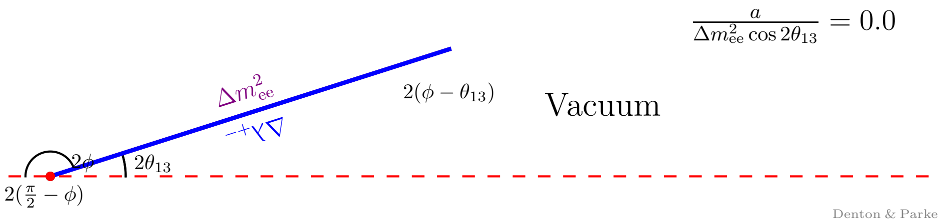
$1 + 2$ Rotations

1. Perform a constant $U_{23}(\theta_{23}, \delta)$ rotation
   - U_{23} commutes with the matter potential
   - Resultant Hamiltonian is real
   - 'Expansion parameter' is $c_{13}s_{13} = 0.15$ at this point
2. Diagonalize the diagonal and $\mathcal{O}(\epsilon^0)$ off-diagonal terms with $U_{13}(\widetilde{\theta}_{13})$
   - $\widetilde{\theta}_{13}(a = 0) = \theta_{13}$
   - Expansion parameter is $c_{12}s_{12}\frac{\Delta m^2_{21}}{\Delta m^2_{ee}} = 0.015$

H. Minakata, S. Parke, 1505.01826

3. Diagonalize the terms non-zero in vacuum with $U_{12}(\widetilde{\theta}_{12})$
   - $\widetilde{\theta}_{12}(a = 0) = \theta_{12}$
   - Expansion parameter is now $\epsilon' = c_{12}s_{12}s_{(\widetilde{\theta}_{13}-\theta_{13})}\frac{\Delta m^2_{21}}{\Delta m^2_{ee}} < 0.015$
   - $\epsilon'(a = 0) = 0$

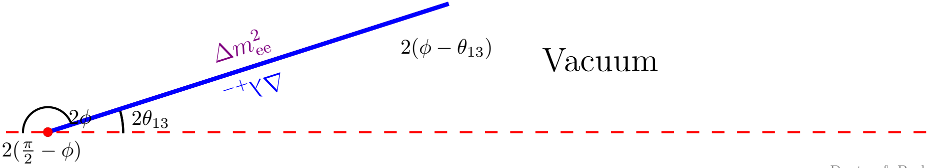
31 Triangle



$\frac{a}{\Delta m^2_{ee}\cos 2\theta_{13}} = 0.0$
Δm^2_{ee}
$2(\phi - \theta_{13})$
Vacuum
2ϕ
$2\theta_{13}$
$2(\frac{\pi}{2} - \phi)$
Denton & Parke

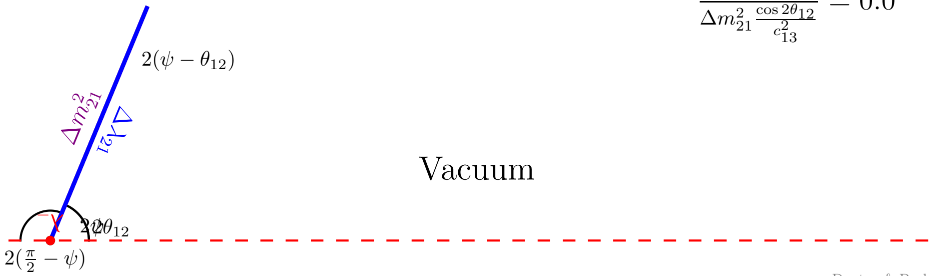
31 Triangle

$$\frac{a}{\Delta m^2_{\rm ee}\cos 2\theta_{13}} = 0.0$$



Denton & Parke

21 Triangle

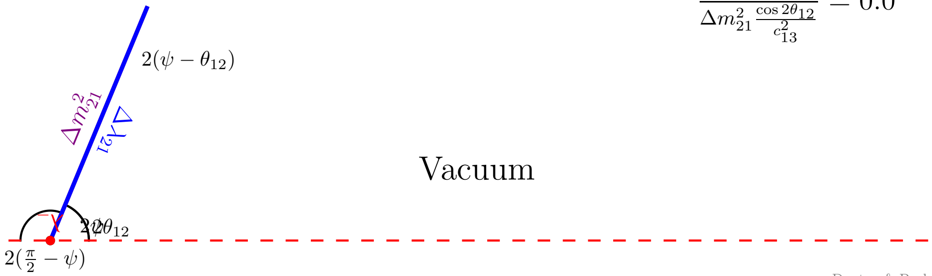


$$\frac{a}{\Delta m^2_{21}\frac{\cos 2\theta_{12}}{c^2_{13}}}=0.0$$

Vacuum

Denton & Parke

21 Triangle



$$\frac{a}{\Delta m^2_{21}\frac{\cos 2\theta_{12}}{c^2_{13}}} = 0.0$$

Vacuum

Denton & Parke

CPV Term

The exact CPV term in matter is

$$P \supset \pm 8 s_\delta c_{12} s_{12} c_{13}^2 s_{13} c_{23} s_{23} \prod_{i>j} \frac{\Delta m_{ij}^2}{\widetilde{\Delta m^2}_{ij}} \sin \widetilde{\Delta_{32}} \sin \widetilde{\Delta_{31}} \sin \widetilde{\Delta_{21}}$$

V. Naumov, Int. J. Mod. Phys. 1992

P. Harrison, W. Scott, hep-ph/9912435

Our expression reproduces this order by order in ϵ' for all channels.

Exact Neutrino Oscillations in Matter: Mixing Angles

$$s^2_{\widehat{12}} = \frac{-\left[(\widehat{m^2}_2)^2 - \alpha\widehat{m^2}_2 + \beta\right]\Delta\widetilde{m^2}_{31}}{\left[(\widehat{m^2}_1)^2 - \alpha\widehat{m^2}_1 + \beta\right]\Delta\widetilde{m^2}_{32} - \left[(\widehat{m^2}_2)^2 - \alpha\widehat{m^2}_2 + \beta\right]\Delta\widetilde{m^2}_{31}}$$

$$s^2_{\widehat{13}} = \frac{(\widehat{m^2}_3)^2 - \alpha\widehat{m^2}_3 + \beta}{\Delta\widetilde{m^2}_{31}\Delta\widetilde{m^2}_{32}}$$

$$s^2_{\widehat{23}} = \frac{s^2_{23}E^2 + c^2_{23}F^2 + 2c_{23}s_{23}c_\delta EF}{E^2 + F^2}$$

$$e^{-i\widehat{\delta}} = \frac{c^2_{23}s^2_{23}\left(e^{-i\delta}E^2 - e^{i\delta}F^2\right) + \left(c^2_{23} - s^2_{23}\right)EF}{\sqrt{\left(s^2_{23}E^2 + c^2_{23}F^2 + 2EFc_{23}s_{23}c_\delta\right)\left(c^2_{23}E^2 + s^2_{23}F^2 - 2EFc_{23}s_{23}c_\delta\right)}}$$

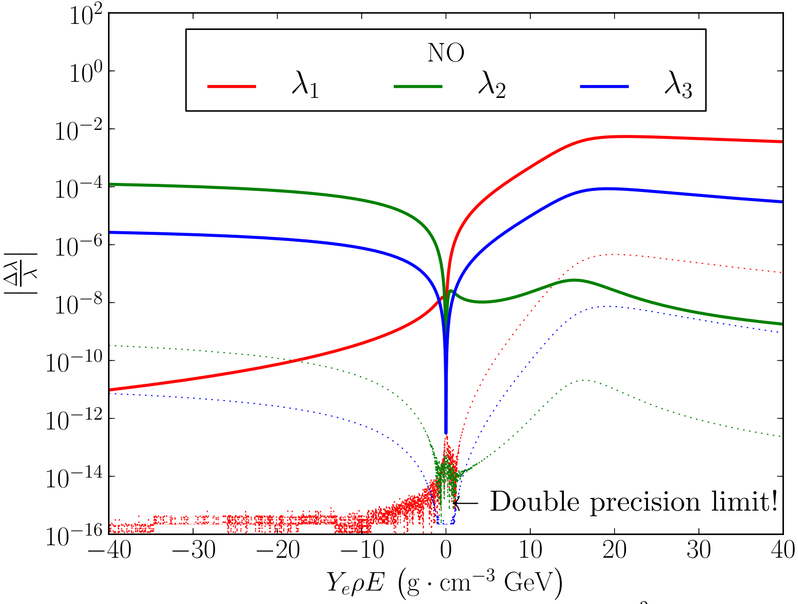
$$\alpha = c^2_{13}\Delta m^2_{31} + \left(c^2_{12}c^2_{13} + s^2_{13}\right)\Delta m^2_{21}\,,\ \ \beta = c^2_{12}c^2_{13}\Delta m^2_{21}\Delta m^2_{31}$$

$$E = c_{13}s_{13}\left[\left(\widehat{m^2}_3 - \Delta m^2_{21}\right)\Delta m^2_{31} - s^2_{12}\left(\widehat{m^2}_3 - \Delta m^2_{31}\right)\Delta m^2_{21}\right]$$

$$F = c_{12}s_{12}c_{13}\left(\widehat{m^2}_3 - \Delta m^2_{31}\right)\Delta m^2_{21}$$

H. Zaglauer, K. Schwarzer, Z. Phys. C Particles and Fields, 40 (1988)

Eigenvalues: Precision



Hamiltonians

After a constant (θ_{23},δ) rotation, $2E\tilde{H}=$

$$\begin{pmatrix}\widetilde{m^2}_a & & s_{13}c_{13}\Delta m^2_{ee}\\ & \widetilde{m^2}_b & \\ s_{13}c_{13}\Delta m^2_{ee} & & \widetilde{m^2}_c\end{pmatrix}+\epsilon s_{12}c_{12}\Delta m^2_{ee}\begin{pmatrix} & c_{13} & \\ c_{13} & & -s_{13}\\ & -s_{13} & \end{pmatrix}$$

After a $U_{13}(\widetilde{\theta}_{13})$ rotation, $2E\hat{H}=$

$$\begin{pmatrix}\widetilde{m^2}_- & & \\ & \widetilde{m^2}_0 & \\ & & \widetilde{m^2}_+\end{pmatrix}+\epsilon c_{12}s_{12}\Delta m^2_{ee}\begin{pmatrix} & c_{(\widetilde{\theta}_{13}-\theta_{13})} & \\ c_{(\widetilde{\theta}_{13}-\theta_{13})} & & s_{(\widetilde{\theta}_{13}-\theta_{13})}\\ & s_{(\widetilde{\theta}_{13}-\theta_{13})} & \end{pmatrix}$$

After a $U_{12}(\widetilde{\theta}_{12})$ rotation, $2E\check{H}=$

$$\begin{pmatrix}\widetilde{m^2}_1 & & \\ & \widetilde{m^2}_2 & \\ & & \widetilde{m^2}_3\end{pmatrix}+\epsilon s_{(\widetilde{\theta}_{13}-\theta_{13})}s_{12}c_{12}\Delta m^2_{ee}\begin{pmatrix} & & -s_{\widetilde{12}}\\ & & c_{\widetilde{12}}\\ -s_{\widetilde{12}} & c_{\widetilde{12}} & \end{pmatrix}$$

Perturbative Expansion

Hamiltonian: $\check{H} = \check{H}_0 + \check{H}_1$

$$\check{H}_0 = \frac{1}{2E}\begin{pmatrix} \widetilde{m^2}_1 & & \\ & \widetilde{m^2}_2 & \\ & & \widetilde{m^2}_3 \end{pmatrix}, \quad \check{H}_1 = \epsilon' \frac{\Delta m^2_{ee}}{2E}\begin{pmatrix} & & -s_{\widetilde{12}} \\ & & c_{\widetilde{12}} \\ -s_{\widetilde{12}} & c_{\widetilde{12}} & \end{pmatrix}$$

Eigenvalues: $\widetilde{m^2}_i^{\text{ex}} = \widetilde{m^2}_i + \widetilde{m^2}_i^{(1)} + \widetilde{m^2}_i^{(2)} + \ldots$

$$\widetilde{m^2}_i^{(1)} = 2E(\check{H}_1)_{ii} = 0$$

$$\widetilde{m^2}_i^{(2)} = \sum_{k\neq i} \frac{[2E(\check{H}_1)_{ik}]^2}{\Delta\widetilde{m^2}_{ik}}$$

Perturbative Expansion: Eigenvectors

Use vacuum expressions with $U \to V$ where

$$V = \widetilde{U} W$$

$\widetilde{U}$ is U with $\theta_{13} \to \widetilde{\theta}_{13}$ and $\theta_{12} \to \widetilde{\theta}_{12}$,

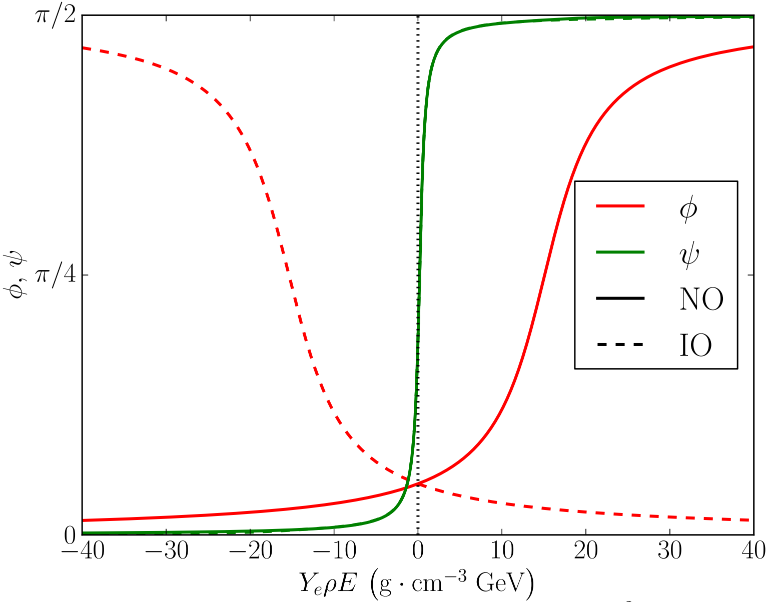
$W = W_0 + W_1 + W_2 + \ldots$

$$W_0 = \mathbb{1}$$

$$W_1 = \epsilon' \Delta m^2_{ee} \begin{pmatrix} & & -\frac{s_{\widetilde{12}}}{\widetilde{\Delta m^2}_{31}} \\ & & \frac{c_{\widetilde{12}}}{\widetilde{\Delta m^2}_{32}} \\ \frac{s_{\widetilde{12}}}{\widetilde{\Delta m^2}_{31}} & -\frac{c_{\widetilde{12}}}{\widetilde{\Delta m^2}_{32}} & \end{pmatrix}$$

$$W_2 = -\epsilon'^2 \frac{(\Delta m^2_{ee})^2}{2} \begin{pmatrix} \frac{s^2_{\widetilde{12}}}{(\widetilde{\Delta m^2}_{31})^2} & -\frac{s_{2\widetilde{12}}}{\widetilde{\Delta m^2}_{32}\widetilde{\Delta m^2}_{21}} & \\ \frac{s_{2\widetilde{12}}}{\widetilde{\Delta m^2}_{31}\widetilde{\Delta m^2}_{21}} & \frac{c^2_{\widetilde{12}}}{(\widetilde{\Delta m^2}_{32})^2} & \\ & & \left[\frac{c^2_{\widetilde{12}}}{(\widetilde{\Delta m^2}_{32})^2} + \frac{s^2_{\widetilde{12}}}{(\widetilde{\Delta m^2}_{31})^2}\right] \end{pmatrix}$$

The Two Matter Angles



$\widetilde{m^2}_{1,2} - \widetilde{\theta}_{12}$ Interchange

From the shape of $U_{12}(\widetilde{\theta}_{12})$, it is clear that the probabilities are invariant under a simultaneous interchange of

$$\widetilde{m^2}_1 \leftrightarrow \widetilde{m^2}_2\,, \qquad \text{and} \qquad \widetilde{\theta}_{12} \to \widetilde{\theta}_{12} \pm \frac{\pi}{2}\,.$$

Since only even powers of $\widetilde{\theta}_{12}$ trig functions $c^2_{\widetilde{12}}, s^2_{\widetilde{12}}, c_{\widetilde{12}}s_{\widetilde{12}}, \cos(2\widetilde{\theta}_{12}), \sin(2\widetilde{\theta}_{12})$ appear in the probabilities, the sign degeneracy is irrelevant.

More usefully, we can write that the probabilities are invariant under the simultaneous interchange of

$$\widetilde{m^2}_1 \leftrightarrow \widetilde{m^2}_2\,, \qquad c^2_{\widetilde{12}} \leftrightarrow s^2_{\widetilde{12}}\,, \qquad \text{and} \qquad c_{\widetilde{12}}s_{\widetilde{12}} \to -c_{\widetilde{12}}s_{\widetilde{12}}\,.$$

This interchange constrains the $\sin^2\Delta_{21}$ term,
and the $\sin^2\Delta_{32}$ term easily follows from the $\sin^2\Delta_{31}$ term.

Zeroth Order Coefficients

$$P_{\alpha\beta} = \delta_{\alpha\beta} + 4C_{21}^{\alpha\beta}\sin^2\Delta_{21} + 4C_{31}^{\alpha\beta}\sin^2\Delta_{31} + 4C_{32}^{\alpha\beta}\sin^2\Delta_{32} + 8D^{\alpha\beta}\sin\Delta_{21}\sin\Delta_{31}\sin\Delta_{32}$$

$\nu_\alpha \to \nu_\beta$	$(C_{21}^{\alpha\beta})^{(0)}$
$\nu_e \to \nu_e$	$-c_{\widetilde{13}}^4 s_{\widetilde{12}}^2 c_{\widetilde{12}}^2$
$\nu_\mu \to \nu_e$	$c_{\widetilde{13}}^2 s_{\widetilde{12}}^2 c_{\widetilde{12}}^2 (c_{23}^2 - s_{\widetilde{13}}^2 s_{23}^2) + c_{2\widetilde{12}} J_r^m c_\delta$
$\nu_\mu \to \nu_\mu$	$-(c_{23}^2 c_{\widetilde{12}}^2 + s_{23}^2 s_{\widetilde{13}}^2 s_{\widetilde{12}}^2)(c_{23}^2 s_{\widetilde{12}}^2 + s_{23}^2 s_{\widetilde{13}}^2 c_{\widetilde{12}}^2)$ $-2(c_{23}^2 - s_{\widetilde{13}}^2 s_{23}^2) c_{2\widetilde{12}} J_{rr}^m c_\delta + (2J_{rr}^m c_\delta)^2$

$\nu_\alpha \to \nu_\beta$	$(C_{31}^{\alpha\beta})^{(0)}$	$(D^{\alpha\beta})^{(0)}$
$\nu_e \to \nu_e$	$-c_{\widetilde{13}}^2 s_{\widetilde{13}}^2 c_{\widetilde{12}}^2$	0
$\nu_\mu \to \nu_e$	$s_{\widetilde{13}}^2 c_{\widetilde{13}}^2 c_{\widetilde{12}}^2 s_{23}^2 + J_r^m c_\delta$	$-J_r^m s_\delta$
$\nu_\mu \to \nu_\mu$	$-c_{\widetilde{13}}^2 s_{23}^2 (c_{23}^2 s_{\widetilde{12}}^2 + s_{23}^2 s_{\widetilde{13}}^2 c_{\widetilde{12}}^2)$ $-2 s_{23}^2 J_r^m c_\delta$	0

$$J_r^m \equiv s_{\widetilde{12}} c_{\widetilde{12}} s_{\widetilde{13}} c_{\widetilde{13}}^2 s_{23} c_{23}, \quad J_{rr}^m \equiv J_r^m / c_{\widetilde{13}}^2$$

General Form of the First Order Coefficients

Can reduce 8 expressions down to 3:

$$(C_{21}^{\alpha\beta})^{(1)} = \epsilon' \Delta m^2_{ee} \left(\frac{F_1^{\alpha\beta}}{\widetilde{\Delta m^2}_{31}} + \frac{F_2^{\alpha\beta}}{\widetilde{\Delta m^2}_{32}} \right)$$

$$(C_{31}^{\alpha\beta})^{(1)} = \epsilon' \Delta m^2_{ee} \left(\frac{F_1^{\alpha\beta} + G_1^{\alpha\beta}}{\widetilde{\Delta m^2}_{31}} - \frac{F_2^{\alpha\beta}}{\widetilde{\Delta m^2}_{32}} \right)$$

$$(C_{32}^{\alpha\beta})^{(1)} = \epsilon' \Delta m^2_{ee} \left(-\frac{F_1^{\alpha\beta}}{\widetilde{\Delta m^2}_{31}} + \frac{F_2^{\alpha\beta} + G_2^{\alpha\beta}}{\widetilde{\Delta m^2}_{32}} \right)$$

$$(D^{\alpha\beta})^{(1)} = \epsilon' \Delta m^2_{ee} \left(\frac{K_1^{\alpha\beta}}{\widetilde{\Delta m^2}_{31}} - \frac{K_2^{\alpha\beta}}{\widetilde{\Delta m^2}_{32}} \right)$$

$$K_1^{\alpha\beta} = \begin{cases} 0 & \alpha = \beta \\ \mp s_{23} c_{23} c_{\widetilde{13}} s^2_{\widetilde{12}} (c^2_{\widetilde{13}} c^2_{\widetilde{12}} - s^2_{\widetilde{13}}) s_\delta & \alpha \neq \beta \end{cases}$$

where the minus sign is for $\nu_\mu \to \nu_e$

First Order Coefficients

$\nu_\alpha \to \nu_\beta$	$F_1^{\alpha\beta}$
$\nu_e \to \nu_e$	$-2c^3_{\widetilde{13}} s_{\widetilde{13}} s^3_{\widetilde{12}} c_{\widetilde{12}}$
$\nu_\mu \to \nu_e$	$c_{\widetilde{13}} s^2_{\widetilde{12}} [s_{\widetilde{13}} s_{\widetilde{12}} c_{\widetilde{12}} (c^2_{23} + c_{2\widetilde{13}} s^2_{23})$ $-s_{23} c_{23} (s^2_{\widetilde{13}} s^2_{\widetilde{12}} + c_{2\widetilde{13}} c^2_{\widetilde{12}}) c_\delta]$
$\nu_\mu \to \nu_\mu$	$2c_{\widetilde{13}} s_{\widetilde{12}} (s^2_{23} s_{\widetilde{13}} c_{\widetilde{12}} + s_{23} c_{23} s_{\widetilde{12}} c_\delta) \times$ $(c^2_{23} c^2_{\widetilde{12}} - 2 s_{23} c_{23} s_{\widetilde{13}} s_{\widetilde{12}} c_{\widetilde{12}} c_\delta + s^2_{23} s^2_{\widetilde{13}} s^2_{\widetilde{12}})$

$\nu_\alpha \to \nu_\beta$	$G_1^{\alpha\beta}$
$\nu_e \to \nu_e$	$2s_{\widetilde{13}} c_{\widetilde{13}} s_{\widetilde{12}} c_{\widetilde{12}} c_{2\widetilde{13}}$
$\nu_\mu \to \nu_e$	$-2s_{\widetilde{13}} c_{\widetilde{13}} s_{\widetilde{12}} (s^2_{23} c_{2\widetilde{13}} c_{\widetilde{12}} - s_{23} c_{23} s_{\widetilde{13}} s_{\widetilde{12}} c_\delta)$
$\nu_\mu \to \nu_\mu$	$-2c_{\widetilde{13}} s_{\widetilde{12}} (s^2_{23} s_{\widetilde{13}} c_{\widetilde{12}} + s_{23} c_{23} s_{\widetilde{12}} c_\delta)$ $\times (1 - 2c^2_{\widetilde{13}} s^2_{23})$

Variable Matter Density

This work assumed ρ is constant

If ρ doesn't vary too much, we can set ρ to the average

$$\rho = \bar{\rho} = \frac{1}{L}\int_0^L \rho(x) dx$$

ρ doesn't vary "too much" when

$$|\dot{\widehat{\theta}}| \ll \left|\frac{\Delta\widehat{m^2}}{2E}\right|$$