Analytic and Compact Expressions for Neutrino Oscillations in Matter

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ICHEP

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Work done with S. Parke, X. Zhang, and H. Minakata.

1604.08167, 1806.01277

github.com/PeterDenton/Nu-Pert

The Niels Bohr International Academy

VILLUM FONDEN

What is $P(\nu_\mu \to \nu_e)$?

$$P(\overset{(-)}{\nu}_\mu \to \overset{(-)}{\nu}_e) = |\mathcal{A}_{\mu e}|^2 \qquad \mathcal{A}_{\mu e} = \mathcal{A}_{31} + e^{\pm i\Delta_{32}}\mathcal{A}_{21}$$

$$\mathcal{A}_{31} = 2s_{13}c_{13}s_{23}\sin\Delta_{31}$$

$$\mathcal{A}_{21} = 2s_{12}c_{13}(c_{12}c_{23}e^{i\delta} - s_{12}s_{13}s_{23})\sin\Delta_{21}$$

$$\Delta_{ij} = \Delta m^2{}_{ij}L/4E$$

...in matter?

Now: NOvA, T2K, MINOS, ...
Upcoming: DUNE, T2HK, ...
Second maximum: T2HKK? ESSnuB? ...

A Simple Solution

For two flavor oscillations:

$$P(\nu_\alpha \to \nu_\beta) = \sin^2 2\theta \sin^2\left(\frac{\Delta m^2 L}{4E}\right)$$

- Solar: θ_{21}, Δm^2_{21}
- Reactor: θ_{13}, Δm^2_{ee}

Neutrino Oscillations Status

Six parameters:

1. $\theta_{13} = 8.5°$
2. $\theta_{12} = 34°$
3. $\Delta m^2_{21} = 7.4 \times 10^{-5}$ eV^2
4. $\theta_{23} \sim 45°$ (octant)
5. $|\Delta m^2_{31}| = 2.5 \times 10^{-3}$ eV^2 (mass ordering)
6. $\delta = ???$

Nu-Fit, 1611.01514

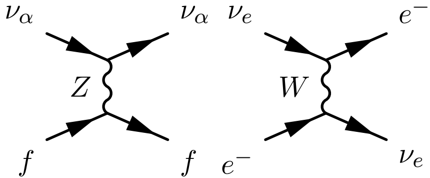
PMNS order allows for easy measurement of θ_{13} and θ_{12}.

Remaining parameters require full three-flavor description.

Matter Effects Matter

$$\mathcal{A}(\nu_\alpha \to \nu_\beta) = \sum_i U^*_{\alpha i} U_{\beta i} e^{-i m_i^2 L/2E} \qquad P = |\mathcal{A}|^2$$

In matter ν's propagate in a new basis that depends on $a \propto \rho E$.



L. Wolfenstein, PRD 17 (1978)

Eigenvalues: $m_i^2 \to \widetilde{m^2}_i(a)$

Eigenvectors are given by $\theta_{ij} \to \widetilde{\theta}_{ij}(a)$ $\Leftarrow$ Unitarity

Hamiltonian Dynamics

$$H = \frac{1}{2E}\left[U\begin{pmatrix}0 & & \\ & \Delta m^2_{21} & \\ & & \Delta m^2_{31}\end{pmatrix}U^\dagger + \begin{pmatrix}a & & \\ & 0 & \\ & & 0\end{pmatrix}\right]$$

$a = 2\sqrt{2}G_F N_e E$

Find eigenvalues and eigenvectors:

$$H = \frac{1}{2E}\widetilde{U}\begin{pmatrix}0 & & \\ & \Delta\widetilde{m^2}_{21} & \\ & & \Delta\widetilde{m^2}_{31}\end{pmatrix}\widetilde{U}^\dagger$$

Computationally works, but we can do better than a **black box**...

Analytic expression?

Eigenvalues Analytically: The Exact Solution

Solve the cubic characteristic equation.

$$\widetilde{m^2}_1 = \frac{A}{3} - \frac{1}{3}\sqrt{A^2-3B}\,S - \frac{\sqrt{3}}{3}\sqrt{A^2-3B}\sqrt{1-S^2}$$

$$\widetilde{m^2}_2 = \frac{A}{3} - \frac{1}{3}\sqrt{A^2-3B}\,S + \frac{\sqrt{3}}{3}\sqrt{A^2-3B}\sqrt{1-S^2}$$

$$\widetilde{m^2}_3 = \frac{A}{3} + \frac{2}{3}\sqrt{A^2-3B}\,S$$

$$A = \Delta m^2_{21} + \Delta m^2_{31} + a$$

$$B = \Delta m^2_{21}\Delta m^2_{31} + a\left[c^2_{13}\Delta m^2_{31} + (c^2_{12}c^2_{13} + s^2_{13})\Delta m^2_{21}\right]$$

$$C = a\Delta m^2_{21}\Delta m^2_{31}c^2_{12}c^2_{13}$$

$$S = \cos\left\{\frac{1}{3}\cos^{-1}\left[\frac{2A^3 - 9AB + 27C}{2\left(A^2-3B\right)^{3/2}}\right]\right\}$$

H. Zaglauer, K. Schwarzer, Z. Phys. C Particles and Fields, 40 (1988)

Traded one **black box** for another...

Alternative Solutions

Perturbative expansion:

- Small matter potential: $a/\Delta m^2$

  Y. Li, Y. Wang, Z-z. Xing, 1605.00900

- s_{13}, s_{13}^2

  A. Cervera, et al., hep-ph/0002108

  H. Minakata, 0910.5545

  K. Asano, H. Minakata, 1103.4387

- $\Delta m_{21}^2/\Delta m_{31}^2 \sim 0.03$

  J. Arafune, J. Sato, hep-ph/9607437

  A. Cervera, et al., hep-ph/0002108

  M. Freund, hep-ph/0103300

  E. Akhmedov, et al., hep-ph/0402175

  M. Blennow, A. Smirnov, 1306.2903

  H. Minakata, S. Parke, 1505.01826

  PBD, H. Minakata, S. Parke, 1604.08167

A Tale of Two Tools

Split the Hamiltonian into:

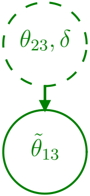
- Large, diagonal part (H_0)
- Small, off-diagonal part (H_1)

- Improves precision at zeroth order
- Naturally leads to using $\Delta m^2_{ee} \equiv c^2_{12}\Delta m^2_{31} + s^2_{12}\Delta m^2_{32}$

H. Nunokawa, S. Parke, R. Zukanovich, hep-ph/0503283

1. **Rotations**:
   - A two-flavor rotation only requires solving a quadratic
   - Diagonalize away the big terms
   - Follows the order of the PMNS matrix
2. **Perturbative expansion**:
   - Smallness parameter is $|\epsilon'| \leq 0.015$
   - Correct eigenvalues and eigenvectors
   - Eigenvalues already include 1^{st} order corrections at 0^{th} order
   - Can improve the precision to arbitrary order

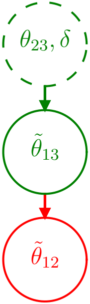
Atmospheric Resonance



1. $U_{23}(\theta_{23}, \delta)$ commutes with matter potential
2. Largest off-diagonal term:
   $s_{13}c_{13}\Delta m^2_{ee}$ in the 1-3 position

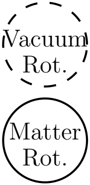
- Eigenvalues still cross at the solar resonance:
  - No perturbation theory there

Solar Resonance



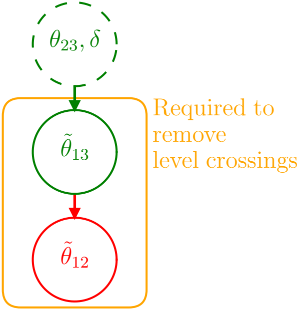
3. Largest off-diagonal term:
   $s_{12}c_{12}c_{\widetilde{\theta}_{13}-\theta_{13}}\Delta m^2_{21}$ in the 1-2 position
   - Largest except for ν's above the atmospheric resonance

- $|\epsilon'| < 0.015$, zero in vacuum
- Perturbation theory valid everywhere now
- Rotation order matches PMNS
- Take vacuum expressions, replace θ_{13}, θ_{12}, and Δm^2_{ij}
- Extremely precise $|\Delta P/P| < 10^{-3}$



MP15
DMP16

Solar Resonance



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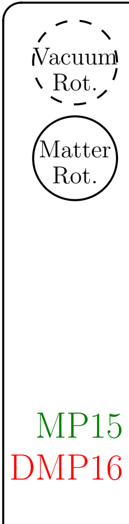
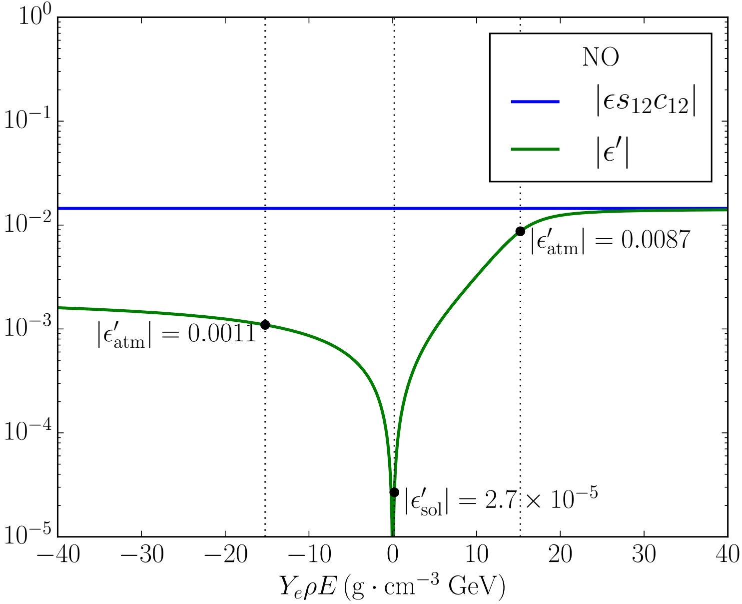


Expansion Parameter



Probability in Matter: DMP 0^{th}

$$\text{Vacuum} \quad \Rightarrow \quad \widetilde{\text{Matter}}$$

$$P_{\alpha\beta}(\Delta m^2_{21}, \Delta m^2_{31}, \theta_{13}, \theta_{12}, \theta_{23}, \delta) \Rightarrow P_{\alpha\beta}(\Delta\widetilde{m^2}_{21}, \Delta\widetilde{m^2}_{31}, \widetilde{\theta}_{13}, \widetilde{\theta}_{12}, \theta_{23}, \delta)$$

Same expression, 4 new variables.

Probability in Matter: DMP 0^{th}

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Same expression, 4 new variables.

$$\cos 2\widetilde{\theta}_{13} = \frac{\Delta m^2_{ee} \cos 2\theta_{13} - a}{\Delta\widetilde{m^2}_{ee}}$$

$$\Delta\widetilde{m^2}_{ee} = \Delta m^2_{ee} \sqrt{(\cos 2\theta_{13} - a/\Delta m^2_{ee})^2 + \sin^2 2\theta_{13}}$$

Probability in Matter: DMP 0^{th}

Vacuum $\Rightarrow$ $\widetilde{\text{Matter}}$

$$P_{\alpha\beta}(\Delta m^2_{21}, \Delta m^2_{31}, \theta_{13}, \theta_{12}, \theta_{23}, \delta) \Rightarrow P_{\alpha\beta}(\Delta\widetilde{m^2}_{21}, \Delta\widetilde{m^2}_{31}, \widetilde{\theta}_{13}, \widetilde{\theta}_{12}, \theta_{23}, \delta)$$

Same expression, 4 new variables.

$$\cos 2\widetilde{\theta}_{13} = \frac{\Delta m^2_{ee}\cos 2\theta_{13} - a}{\Delta\widetilde{m^2}_{ee}}$$

$$\Delta\widetilde{m^2}_{ee} = \Delta m^2_{ee}\sqrt{(\cos 2\theta_{13} - a/\Delta m^2_{ee})^2 + \sin^2 2\theta_{13}}$$

$$\cos 2\widetilde{\theta}_{12} = \frac{\Delta m^2_{21}\cos 2\theta_{12} - a_{12}}{\Delta\widetilde{m^2}_{21}}, \qquad a_{12} = (a + \Delta m^2_{ee} - \Delta\widetilde{m^2}_{ee})/2$$

$$\Delta\widetilde{m^2}_{21} = \Delta m^2_{21}\sqrt{(\cos 2\theta_{12} - a_{12}/\Delta m^2_{21})^2 + \cos^2(\widetilde{\theta}_{13} - \theta_{13})\sin^2 2\theta_{12}}$$

Probability in Matter: DMP 0^{th}

$$\text{Vacuum} \Rightarrow \widetilde{\text{Matter}}$$

$$P_{\alpha\beta}(\Delta m^2_{21}, \Delta m^2_{31}, \theta_{13}, \theta_{12}, \theta_{23}, \delta) \Rightarrow P_{\alpha\beta}(\widetilde{\Delta m^2}_{21}, \widetilde{\Delta m^2}_{31}, \widetilde{\theta}_{13}, \widetilde{\theta}_{12}, \theta_{23}, \delta)$$

Same expression, 4 new variables.

$$\cos 2\widetilde{\theta}_{13} = \frac{\Delta m^2_{ee}\cos 2\theta_{13} - a}{\widetilde{\Delta m^2}_{ee}}$$

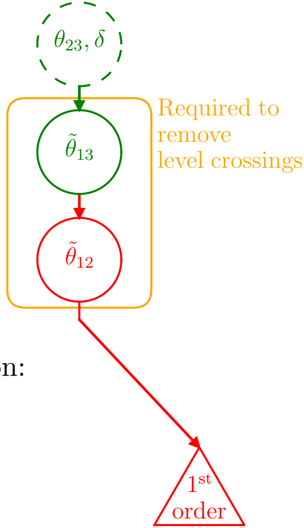
$$\widetilde{\Delta m^2}_{ee} = \Delta m^2_{ee}\sqrt{(\cos 2\theta_{13} - a/\Delta m^2_{ee})^2 + \sin^2 2\theta_{13}}$$

$$\cos 2\widetilde{\theta}_{12} = \frac{\Delta m^2_{21}\cos 2\theta_{12} - a_{12}}{\widetilde{\Delta m^2}_{21}}, \qquad a_{12} = (a + \Delta m^2_{ee} - \widetilde{\Delta m^2}_{ee})/2$$

$$\widetilde{\Delta m^2}_{21} = \Delta m^2_{21}\sqrt{(\cos 2\theta_{12} - a_{12}/\Delta m^2_{21})^2 + \cos^2(\widetilde{\theta}_{13} - \theta_{13})\sin^2 2\theta_{12}}$$

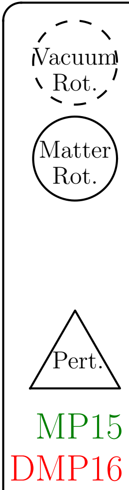
$$\widetilde{\Delta m^2}_{31} = \Delta m^2_{31} + \frac{1}{4}a + \frac{1}{2}(\widetilde{\Delta m^2}_{21} - \Delta m^2_{21}) + \frac{3}{4}(\widetilde{\Delta m^2}_{ee} - \Delta m^2_{ee})$$

Improve with Perturbation

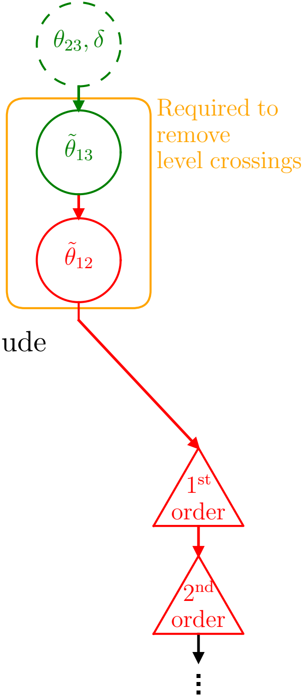


4. $\gtrsim 2$ orders of magnitude of improvement in precision: $|\Delta P/P| < 10^{-6}$

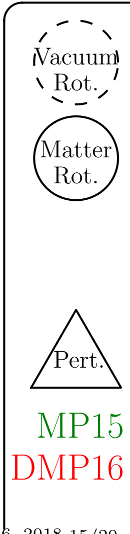
- Eigenvalues need no correction
- Compact form utilizes a $\widetilde{m^2}_1 \leftrightarrow \widetilde{m^2}_2$, $\widetilde{\theta}_{12} \leftrightarrow \widetilde{\theta}_{12} \pm \pi/2$ symmetry



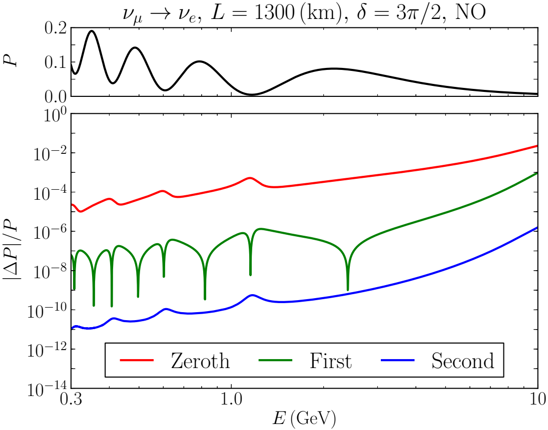
Higher Orders



5. $\gtrsim 2$ more orders of magnitude of improvement per order: $|\Delta P/P| < 10^{-9}$, …



Precision



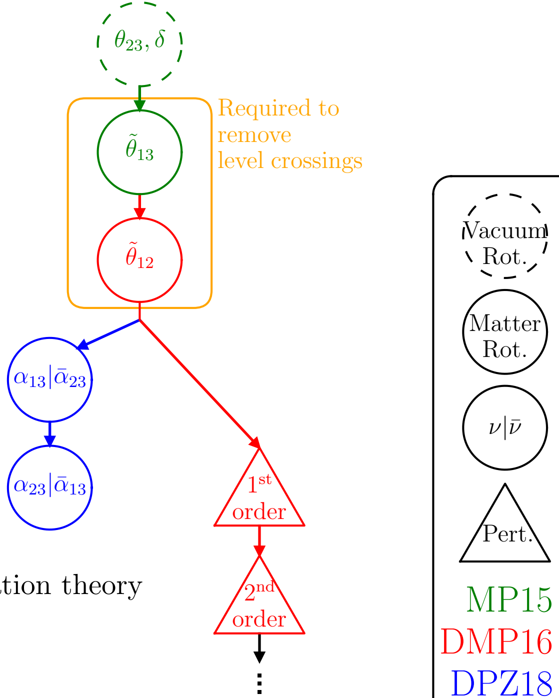
DUNE: NO, $\delta = 3\pi/2$		First min	First max
$P(\nu_\mu \to \nu_e)$		0.0047	0.081
E (GeV)		1.2	2.2
$\frac{\lvert\Delta P\rvert}{P}$	Zeroth	5×10^{-4}	4×10^{-4}
	First	3×10^{-7}	2×10^{-7}
	Second	6×10^{-10}	5×10^{-10}

More Rotations

Instead continue to diagonalize large terms

4. 1-3 sector for ν's
   2-3 sector for $\bar{\nu}$'s
5. Then opposite

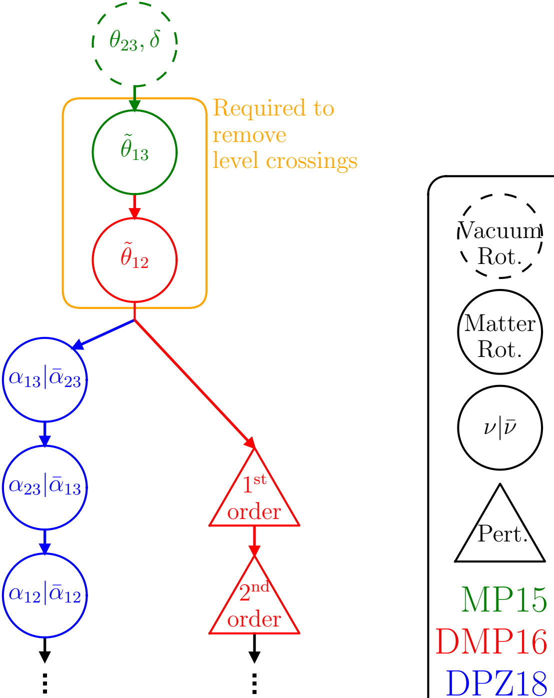
- 2 additional rotations
  $\equiv$ 1 order of perturbation theory



Even More Rotations

6. 1-2 sector for either $\nu/\bar{\nu}$

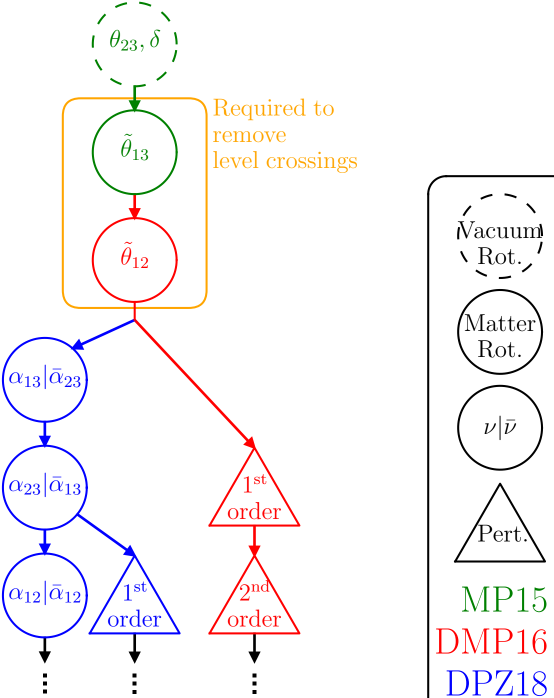
- 3 additional rotations
  $\equiv$ 2 orders of pert. th.



More Options

6. Perturbation theory after 2 additional rotations

- 2 additional rotations + 1 order of pert. th. ≡ 2 orders of pert. th.



Key Points

- Include 1$^{\text{st}}$ order corrections in 0$^{\text{th}}$ order eigenvalues (Δm^2_{ee})
- Rotate **large terms first** $\Rightarrow$ PMNS order, removes level crossings
- All channels, energies, and baselines handled simultaneously
- 0$^{\text{th}}$ order probabilities: **same structure as vacuum** probabilities
- 0$^{\text{th}}$ order: **accurate** enough for current & future experiments
- **Further precision** through perturbation and/or more rotations

Backups

Neutrino Oscillations in Vacuum: Disappearance

It is easy to calculate the *exact* disappearance expression in vacuum:

$$P(\nu_\alpha \to \nu_\alpha) = 1 - 4\sum_{i<j} |U_{\alpha i}|^2 |U_{\alpha j}|^2 \sin^2 \Delta_{ji}$$

For the electron case this expression is simple:

$$\begin{aligned} P(\nu_e \to \nu_e) = 1 & \\ & - 4c_{12}^2 s_{12}^2 c_{13}^4 \sin^2 \Delta_{21} \\ & - 4c_{12}^2 c_{13}^2 s_{13}^2 \sin^2 \Delta_{31} \\ & - 4s_{12}^2 c_{13}^2 s_{13}^2 \sin^2 \Delta_{32} \end{aligned}$$

$$\Delta_{ij} = \frac{\Delta m_{ij}^2 L}{4E}$$
$$\Delta m_{ij}^2 = m_i^2 - m_j^2$$

Alternative Solutions: Example

$$P_0 = \sin^2\theta_{23}\frac{\sin^2 2\theta_{13}}{\hat{C}^2}\sin^2(\hat{\Delta}\hat{C}), \tag{36a}$$

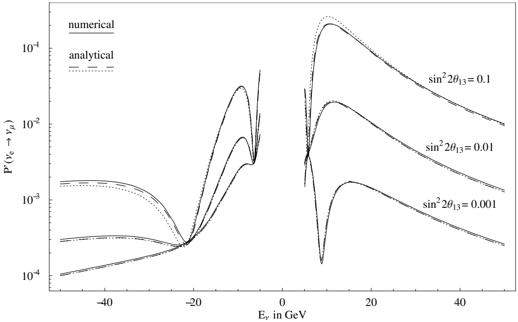
$$P_{\sin\delta} = \frac{1}{2}\alpha\frac{\sin\delta\cos\theta_{13}\sin 2\theta_{12}\sin 2\theta_{13}\sin 2\theta_{23}}{\hat{A}\hat{C}\cos\theta_{13}^2}\sin(\hat{C}\hat{\Delta}) \times\{\cos(\hat{C}\hat{\Delta}) - \cos((1+\hat{A})\hat{\Delta})\}, \tag{36b}$$

$$P_{\cos\delta} = \frac{1}{2}\alpha\frac{\cos\delta\cos\theta_{13}\sin 2\theta_{12}\sin 2\theta_{13}\sin 2\theta_{23}}{\hat{A}\hat{C}\cos\theta_{13}^2}\sin(\hat{C}\hat{\Delta}) \times\{\sin((1+\hat{A})\hat{\Delta}) \mp \sin(\hat{C}\hat{\Delta})\}, \tag{36c}$$

$$P_1 = -\alpha\frac{1-\hat{A}\cos 2\theta_{13}}{\hat{C}^3}\sin^2\theta_{12}\sin^2 2\theta_{13}\sin^2\theta_{23}\hat{\Delta} \times\sin(2\hat{\Delta}\hat{C}) + \alpha\frac{2\hat{A}(-\hat{A}+\cos 2\theta_{13})}{\hat{C}^4} \times\sin^2\theta_{12}\sin^2 2\theta_{13}\sin^2\theta_{23}\sin^2(\hat{\Delta}\hat{C}), \tag{36d}$$

$$P_2 = \alpha\frac{\mp 1 + \hat{C} \pm \hat{A}\cos 2\theta_{13}}{2\hat{C}^2\hat{A}\cos^2\theta_{13}}\cos\theta_{13}\sin 2\theta_{12}\sin 2\theta_{13} \times\sin 2\theta_{23}\sin^2(\hat{\Delta}\hat{C}), \tag{36e}$$

$$P_3 = \alpha^2\frac{2\hat{C}\cos^2\theta_{23}\sin^2 2\theta_{12}}{\hat{A}^2\cos^2\theta_{13}(\mp\hat{A}+\hat{C}\pm\cos 2\theta_{13})} \times\sin^2\left(\frac{1}{2}(1+\hat{A}\mp\hat{C})\hat{\Delta}\right). \tag{36f}$$



M. Freund, hep-ph/0103300

Our Methodology

- Start with $\epsilon = \frac{\Delta m^2_{21}}{\Delta m^2_{ee}} = 0.03$
- Perform one fixed and two variable rotations: (θ_{23}, δ), $\widetilde{\theta}_{13}$, $\widetilde{\theta}_{12}$
- Write the probabilities with simple L/E dependence:

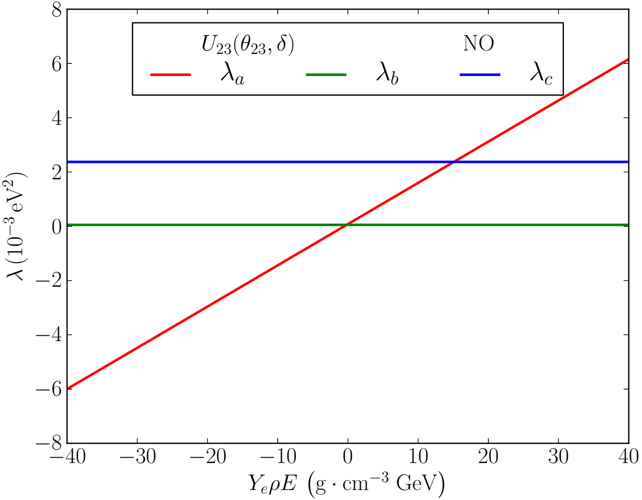
$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - \sum_{i<j} \Re\left[U_{\alpha i} U^*_{\beta i} U^*_{\alpha j} U_{\beta j}\right] \sin^2 \Delta_{ij}$$
$$+ 8\Im\left[U_{\alpha 1} U^*_{\beta 2} U^*_{\alpha 2} U_{\beta 1}\right] \sin\Delta_{32} \sin\Delta_{31} \sin\Delta_{21}$$

C. Jarlskog: PRL 55 (1985)

Nonvanishing Wronskian $\Rightarrow$ fewest number of L/E functions

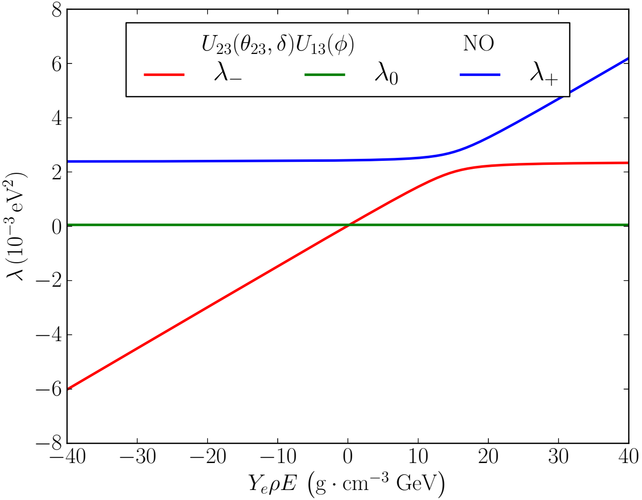
Clear that the CPV term is $\mathcal{O}[(L/E)^3]$ not $\mathcal{O}[(L/E)^1]$

Eigenvalues in Matter: Two Rotations are Needed



$$\widetilde{m^2}_a = a + (s_{13}^2 + \epsilon s_{12}^2)\Delta m_{ee}^2\,,\ \widetilde{m^2}_b = \epsilon c_{12}^2 \Delta m_{ee}^2\,,\ \widetilde{m^2}_c = (c_{13}^2 + \epsilon s_{12}^2)\Delta m_{ee}^2$$

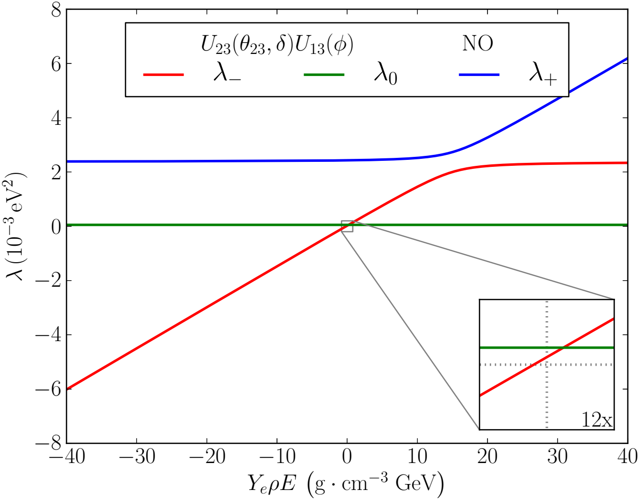
Eigenvalues in Matter: Two Rotations are Needed



$$\widetilde{m^2}_{\mp} = \frac{1}{2}\left[(\widetilde{m^2}_a + \widetilde{m^2}_c) \mp \mathrm{sgn}(\Delta m^2_{ee})\sqrt{(\widetilde{m^2}_c - \widetilde{m^2}_a)^2 + (2s_{13}c_{13}\Delta m^2_{ee})^2}\right]$$

$$\widetilde{m^2}_0 = \widetilde{m^2}_b$$

Eigenvalues in Matter: Two Rotations are Needed



$$\widetilde{m^2}_{\mp} = \frac{1}{2}\left[(\widetilde{m^2}_a + \widetilde{m^2}_c) \mp \mathrm{sgn}(\Delta m^2_{ee})\sqrt{(\widetilde{m^2}_c - \widetilde{m^2}_a)^2 + (2s_{13}c_{13}\Delta m^2_{ee})^2}\right]$$

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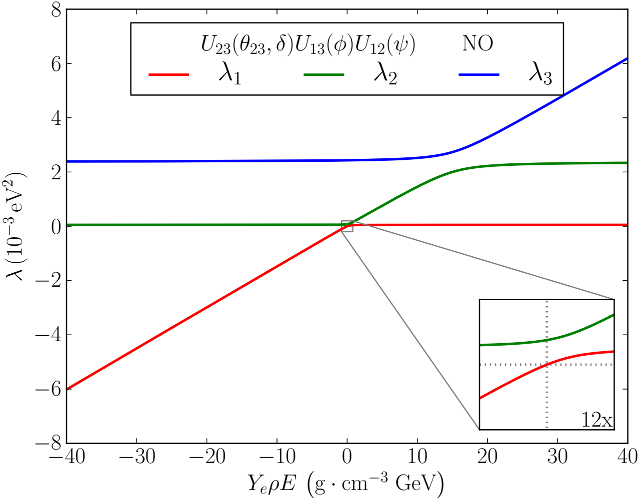
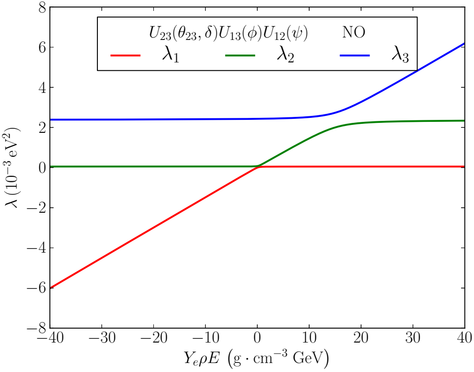
Eigenvalues in Matter: Two Rotations are Needed

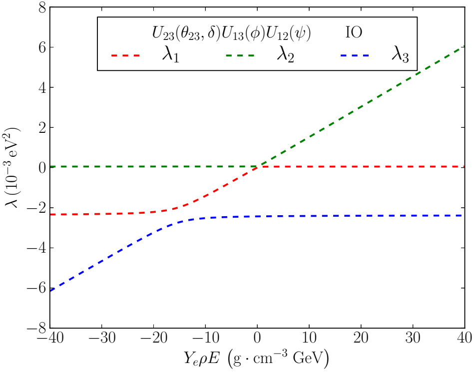


$$\widetilde{m^2}_{1,2} = \frac{1}{2}\left[(\widetilde{m^2}_0 + \widetilde{m^2}_-) \mp \sqrt{(\widetilde{m^2}_0 - \widetilde{m^2}_-)^2 + (2\epsilon c_{(\widetilde{\theta}_{13}-\theta_{13})} c_{12} s_{12} \Delta m^2_{ee})^2}\right]$$

$$\widetilde{m^2}_3 = \widetilde{m^2}_+$$

Eigenvalues in Matter: Mass Ordering





NO

$$\widetilde{m^2}_1 < \widetilde{m^2}_2 < \widetilde{m^2}_3$$

IO

$$\widetilde{m^2}_3 < \widetilde{m^2}_1 < \widetilde{m^2}_2$$

$1 + 2$ Rotations

1. Perform a constant $U_{23}(\theta_{23}, \delta)$ rotation
   - U_{23} commutes with the matter potential
   - Resultant Hamiltonian is real
   - 'Expansion parameter' is $c_{13}s_{13} = 0.15$ at this point
2. Diagonalize the diagonal and $\mathcal{O}(\epsilon^0)$ off-diagonal terms with $U_{13}(\widetilde{\theta}_{13})$
   - $\widetilde{\theta}_{13}(a = 0) = \theta_{13}$
   - Expansion parameter is $c_{12}s_{12}\frac{\Delta m^2_{21}}{\Delta m^2_{ee}} = 0.015$

H. Minakata, S. Parke, 1505.01826

3. Diagonalize the terms non-zero in vacuum with $U_{12}(\widetilde{\theta}_{12})$
   - $\widetilde{\theta}_{12}(a = 0) = \theta_{12}$
   - Expansion parameter is now $\epsilon' = c_{12}s_{12}s_{(\widetilde{\theta}_{13}-\theta_{13})}\frac{\Delta m^2_{21}}{\Delta m^2_{ee}} < 0.015$
   - $\epsilon'(a = 0) = 0$

CPV Term

The exact CPV term in matter is

$$P \supset \pm 8 s_\delta c_{12} s_{12} c_{13}^2 s_{13} c_{23} s_{23} \prod_{i>j} \frac{\Delta m_{ij}^2}{\widetilde{\Delta m^2}_{ij}} \sin\Delta_{32}^m \sin\Delta_{31}^m \sin\Delta_{21}^m$$

V. Naumov, Int. J. Mod. Phys. 1992

P. Harrison, W. Scott, hep-ph/9912435

Our expression reproduces this order by order in ϵ' for all channels.

Exact Neutrino Oscillations in Matter: Mixing Angles

$$s^2_{\widetilde{12}} = \frac{-\left[(\widetilde{m^2}_2)^2 - \alpha\widetilde{m^2}_2 + \beta\right]\Delta\widetilde{m^2}_{31}}{\left[(\widetilde{m^2}_1)^2 - \alpha\widetilde{m^2}_1 + \beta\right]\Delta\widetilde{m^2}_{32} - \left[(\widetilde{m^2}_2)^2 - \alpha\widetilde{m^2}_2 + \beta\right]\Delta\widetilde{m^2}_{31}}$$

$$s^2_{\widetilde{13}} = \frac{(\widetilde{m^2}_3)^2 - \alpha\widetilde{m^2}_3 + \beta}{\Delta\widetilde{m^2}_{31}\Delta\widetilde{m^2}_{32}}$$

$$s^2_{\widetilde{23}} = \frac{s^2_{23}E^2 + c^2_{23}F^2 + 2c_{23}s_{23}c_\delta EF}{E^2 + F^2}$$

$$e^{-i\widetilde{\delta}} = \frac{c^2_{23}s^2_{23}\left(e^{-i\delta}E^2 - e^{i\delta}F^2\right) + \left(c^2_{23} - s^2_{23}\right)EF}{\sqrt{\left(s^2_{23}E^2 + c^2_{23}F^2 + 2EFc_{23}s_{23}c_\delta\right)\left(c^2_{23}E^2 + s^2_{23}F^2 - 2EFc_{23}s_{23}c_\delta\right)}}$$

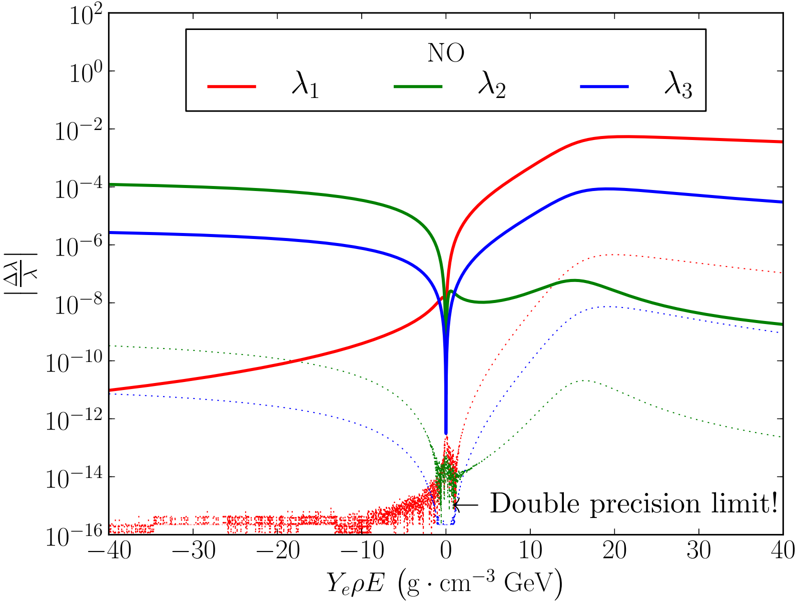
$$\alpha = c^2_{13}\Delta m^2_{31} + \left(c^2_{12}c^2_{13} + s^2_{13}\right)\Delta m^2_{21}\,, \quad \beta = c^2_{12}c^2_{13}\Delta m^2_{21}\Delta m^2_{31}$$

$$E = c_{13}s_{13}\left[\left(\widetilde{m^2}_3 - \Delta m^2_{21}\right)\Delta m^2_{31} - s^2_{12}\left(\widetilde{m^2}_3 - \Delta m^2_{31}\right)\Delta m^2_{21}\right]$$

$$F = c_{12}s_{12}c_{13}\left(\widetilde{m^2}_3 - \Delta m^2_{31}\right)\Delta m^2_{21}$$

H. Zaglauer, K. Schwarzer, Z. Phys. C Particles and Fields, 40 (1988)

Eigenvalues: Precision



Hamiltonians

After a constant (θ_{23}, δ) rotation, $2E\tilde{H} =$

$$\begin{pmatrix} \widetilde{m^2}_a & & s_{13}c_{13}\Delta m^2_{ee} \\ & \widetilde{m^2}_b & \\ s_{13}c_{13}\Delta m^2_{ee} & & \widetilde{m^2}_c \end{pmatrix} + \epsilon s_{12}c_{12}\Delta m^2_{ee} \begin{pmatrix} & c_{13} & \\ c_{13} & & -s_{13} \\ & -s_{13} & \end{pmatrix}$$

After a $U_{13}(\widetilde{\theta}_{13})$ rotation, $2E\hat{H} =$

$$\begin{pmatrix} \widetilde{m^2}_- & & \\ & \widetilde{m^2}_0 & \\ & & \widetilde{m^2}_+ \end{pmatrix} + \epsilon c_{12}s_{12}\Delta m^2_{ee} \begin{pmatrix} & c_{(\widetilde{\theta}_{13}-\theta_{13})} & \\ c_{(\widetilde{\theta}_{13}-\theta_{13})} & & s_{(\widetilde{\theta}_{13}-\theta_{13})} \\ & s_{(\widetilde{\theta}_{13}-\theta_{13})} & \end{pmatrix}$$

After a $U_{12}(\widetilde{\theta}_{12})$ rotation, $2E\check{H} =$

$$\begin{pmatrix} \widetilde{m^2}_1 & & \\ & \widetilde{m^2}_2 & \\ & & \widetilde{m^2}_3 \end{pmatrix} + \epsilon s_{(\widetilde{\theta}_{13}-\theta_{13})} s_{12}c_{12}\Delta m^2_{ee} \begin{pmatrix} & & -s_{\widetilde{12}} \\ & & c_{\widetilde{12}} \\ -s_{\widetilde{12}} & c_{\widetilde{12}} & \end{pmatrix}$$

Perturbative Expansion

Hamiltonian: $\check{H} = \check{H}_0 + \check{H}_1$

$$\check{H}_0 = \frac{1}{2E}\begin{pmatrix} \widetilde{m^2}_1 & & \\ & \widetilde{m^2}_2 & \\ & & \widetilde{m^2}_3 \end{pmatrix}, \quad \check{H}_1 = \epsilon' \frac{\Delta m^2_{ee}}{2E}\begin{pmatrix} & & -s_{\widetilde{12}} \\ & & c_{\widetilde{12}} \\ -s_{\widetilde{12}} & c_{\widetilde{12}} & \end{pmatrix}$$

Eigenvalues: $\widetilde{m^2}_i^{\text{ex}} = \widetilde{m^2}_i + \widetilde{m^2}_i^{(1)} + \widetilde{m^2}_i^{(2)} + \dots$

$$\widetilde{m^2}_i^{(1)} = 2E(\check{H}_1)_{ii} = 0$$

$$\widetilde{m^2}_i^{(2)} = \sum_{k\neq i}\frac{[2E(\check{H}_1)_{ik}]^2}{\Delta\widetilde{m^2}_{ik}}$$

Perturbative Expansion: Eigenvectors

Use vacuum expressions with $U \to V$ where

$$V = U^m W$$

U^m is U with $\theta_{13} \to \widetilde{\theta}_{13}$ and $\theta_{12} \to \widetilde{\theta}_{12}$,

$W = W_0 + W_1 + W_2 + \dots$

$$W_0 = \mathbb{1}$$

$$W_1 = \epsilon' \Delta m^2_{ee} \begin{pmatrix} & & -\frac{s_{\widetilde{12}}}{\widetilde{\Delta m^2}_{31}} \\ & & \frac{c_{\widetilde{12}}}{\widetilde{\Delta m^2}_{32}} \\ \frac{s_{\widetilde{12}}}{\widetilde{\Delta m^2}_{31}} & -\frac{c_{\widetilde{12}}}{\widetilde{\Delta m^2}_{32}} & \end{pmatrix}$$

$$W_2 = -\epsilon'^2 \frac{(\Delta m^2_{ee})^2}{2} \begin{pmatrix} \frac{s^2_{\widetilde{12}}}{(\widetilde{\Delta m^2}_{31})^2} & -\frac{s_{2\widetilde{12}}}{\widetilde{\Delta m^2}_{32}\widetilde{\Delta m^2}_{21}} & \\ \frac{s_{2\widetilde{12}}}{\widetilde{\Delta m^2}_{31}\widetilde{\Delta m^2}_{21}} & \frac{c^2_{\widetilde{12}}}{(\widetilde{\Delta m^2}_{32})^2} & \\ & & \left[\frac{c^2_{\widetilde{12}}}{(\widetilde{\Delta m^2}_{32})^2} + \frac{s^2_{\widetilde{12}}}{(\widetilde{\Delta m^2}_{31})^2}\right] \end{pmatrix}$$

$\widetilde{m^2}_{1,2} - \widetilde{\theta}_{12}$ Interchange

From the shape of $U_{12}(\widetilde{\theta}_{12})$, it is clear that the probabilities are invariant under a simultaneous interchange of

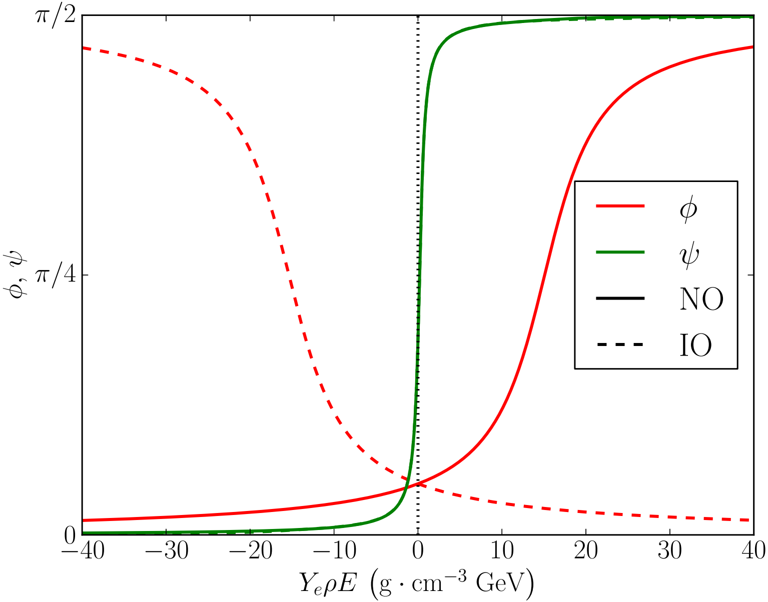
$$\widetilde{m^2}_1 \leftrightarrow \widetilde{m^2}_2\,, \qquad \text{and} \qquad \widetilde{\theta}_{12} \to \widetilde{\theta}_{12} \pm \frac{\pi}{2}\,.$$

Since only even powers of $\widetilde{\theta}_{12}$ trig functions ($c^2_{\widetilde{12}}, s^2_{\widetilde{12}}, c_{\widetilde{12}}s_{\widetilde{12}}, c_{2\widetilde{12}}, s_{2\widetilde{12}}$) appear in the probabilities, the sign degeneracy is irrelevant.
More usefully, we can write that the probabilities are invariant under the simultaneous interchange of

$$\widetilde{m^2}_1 \leftrightarrow \widetilde{m^2}_2\,, \qquad c^2_{\widetilde{12}} \leftrightarrow s^2_{\widetilde{12}}\,, \qquad \text{and} \qquad c_{\widetilde{12}}s_{\widetilde{12}} \to -c_{\widetilde{12}}s_{\widetilde{12}}\,.$$

This interchange constrains C_{21},
and C_{32} is then easily calculated from C_{31}.

The Two Matter Angles



Zeroth Order Coefficients

$$P_{\alpha\beta} = \delta_{\alpha\beta} + 4C_{21}^{\alpha\beta}\sin^2\Delta_{21} + 4C_{31}^{\alpha\beta}\sin^2\Delta_{31} + 4C_{32}^{\alpha\beta}\sin^2\Delta_{32} + 8D^{\alpha\beta}\sin\Delta_{21}\sin\Delta_{31}\sin\Delta_{32}$$

$\nu_\alpha \to \nu_\beta$	$(C_{21}^{\alpha\beta})^{(0)}$
$\nu_e \to \nu_e$	$-c_{\widetilde{13}}^4 s_{\widetilde{12}}^2 c_{\widetilde{12}}^2$
$\nu_\mu \to \nu_e$	$c_{\widetilde{13}}^2 s_{\widetilde{12}}^2 c_{\widetilde{12}}^2 (c_{23}^2 - s_{\widetilde{13}}^2 s_{23}^2) + c_{2\widetilde{12}} J_r^m c_\delta$
$\nu_\mu \to \nu_\mu$	$-(c_{23}^2 c_{\widetilde{12}}^2 + s_{23}^2 s_{\widetilde{13}}^2 s_{\widetilde{12}}^2)(c_{23}^2 s_{\widetilde{12}}^2 + s_{23}^2 s_{\widetilde{13}}^2 c_{\widetilde{12}}^2)$ $-2(c_{23}^2 - s_{\widetilde{13}}^2 s_{23}^2) c_{2\widetilde{12}} J_{rr}^m c_\delta + (2J_{rr}^m c_\delta)^2$

$\nu_\alpha \to \nu_\beta$	$(C_{31}^{\alpha\beta})^{(0)}$	$(D^{\alpha\beta})^{(0)}$
$\nu_e \to \nu_e$	$-c_{\widetilde{13}}^2 s_{\widetilde{13}}^2 c_{\widetilde{12}}^2$	0
$\nu_\mu \to \nu_e$	$s_{\widetilde{13}}^2 c_{\widetilde{13}}^2 c_{\widetilde{12}}^2 s_{23}^2 + J_r^m c_\delta$	$-J_r^m s_\delta$
$\nu_\mu \to \nu_\mu$	$-c_{\widetilde{13}}^2 s_{23}^2 (c_{23}^2 s_{\widetilde{12}}^2 + s_{23}^2 s_{\widetilde{13}}^2 c_{\widetilde{12}}^2)$ $-2 s_{23}^2 J_r^m c_\delta$	0

$$J_r^m \equiv s_{\widetilde{12}} c_{\widetilde{12}} s_{\widetilde{13}} c_{\widetilde{13}}^2 s_{23} c_{23}, \quad J_{rr}^m \equiv J_r^m / c_{\widetilde{13}}^2$$

General Form of the First Order Coefficients

Can reduce 8 expressions down to 3:

$$(C_{21}^{\alpha\beta})^{(1)} = \epsilon' \Delta m^2_{ee} \left(\frac{F_1^{\alpha\beta}}{\widetilde{\Delta m^2}_{31}} + \frac{F_2^{\alpha\beta}}{\widetilde{\Delta m^2}_{32}} \right)$$

$$(C_{31}^{\alpha\beta})^{(1)} = \epsilon' \Delta m^2_{ee} \left(\frac{F_1^{\alpha\beta} + G_1^{\alpha\beta}}{\widetilde{\Delta m^2}_{31}} - \frac{F_2^{\alpha\beta}}{\widetilde{\Delta m^2}_{32}} \right)$$

$$(C_{32}^{\alpha\beta})^{(1)} = \epsilon' \Delta m^2_{ee} \left(-\frac{F_1^{\alpha\beta}}{\widetilde{\Delta m^2}_{31}} + \frac{F_2^{\alpha\beta} + G_2^{\alpha\beta}}{\widetilde{\Delta m^2}_{32}} \right)$$

$$(D^{\alpha\beta})^{(1)} = \epsilon' \Delta m^2_{ee} \left(\frac{K_1^{\alpha\beta}}{\widetilde{\Delta m^2}_{31}} - \frac{K_2^{\alpha\beta}}{\widetilde{\Delta m^2}_{32}} \right)$$

$$K_1^{\alpha\beta} = \begin{cases} 0 & \alpha = \beta \\ \mp s_{23} c_{23} c_{\widetilde{13}} s^2_{\widetilde{12}} (c^2_{\widetilde{13}} c^2_{\widetilde{12}} - s^2_{\widetilde{13}}) s_\delta & \alpha \neq \beta \end{cases}$$

where the minus sign is for $\nu_\mu \to \nu_e$

First Order Coefficients

$\nu_\alpha \to \nu_\beta$	$F_1^{\alpha\beta}$
$\nu_e \to \nu_e$	$-2c_{\widetilde{13}}^3 s_{\widetilde{13}} s_{\widetilde{12}}^3 c_{\widetilde{12}}$
$\nu_\mu \to \nu_e$	$c_{\widetilde{13}} s_{\widetilde{12}}^2 [s_{\widetilde{13}} s_{\widetilde{12}} c_{\widetilde{12}} (c_{23}^2 + c_{2\widetilde{13}} s_{23}^2)$ $-s_{23} c_{23} (s_{\widetilde{13}}^2 s_{\widetilde{12}}^2 + c_{2\widetilde{13}} c_{\widetilde{12}}^2) c_\delta]$
$\nu_\mu \to \nu_\mu$	$2c_{\widetilde{13}} s_{\widetilde{12}} (s_{23}^2 s_{\widetilde{13}} c_{\widetilde{12}} + s_{23} c_{23} s_{\widetilde{12}} c_\delta) \times$ $(c_{23}^2 c_{\widetilde{12}}^2 - 2 s_{23} c_{23} s_{\widetilde{13}} s_{\widetilde{12}} c_{\widetilde{12}} c_\delta + s_{23}^2 s_{\widetilde{13}}^2 s_{\widetilde{12}}^2)$

$\nu_\alpha \to \nu_\beta$	$G_1^{\alpha\beta}$
$\nu_e \to \nu_e$	$2 s_{\widetilde{13}} c_{\widetilde{13}} s_{\widetilde{12}} c_{\widetilde{12}} c_{2\widetilde{13}}$
$\nu_\mu \to \nu_e$	$-2 s_{\widetilde{13}} c_{\widetilde{13}} s_{\widetilde{12}} (s_{23}^2 c_{2\widetilde{13}} c_{\widetilde{12}} - s_{23} c_{23} s_{\widetilde{13}} s_{\widetilde{12}} c_\delta)$
$\nu_\mu \to \nu_\mu$	$-2 c_{\widetilde{13}} s_{\widetilde{12}} (s_{23}^2 s_{\widetilde{13}} c_{\widetilde{12}} + s_{23} c_{23} s_{\widetilde{12}} c_\delta)$ $\times (1 - 2 c_{\widetilde{13}}^2 s_{23}^2)$

Variable Matter Density

This work assumed ρ is constant

If ρ doesn't vary too much, we can set ρ to the average

$$\rho = \bar{\rho} = \frac{1}{L}\int_0^L \rho(x)dx$$

ρ doesn't vary "too much" when

$$|\dot{\theta}^m| \ll \left|\frac{\widetilde{\Delta m^2}}{2E}\right|$$