Abstract

We further develop and extend a recent perturbative framework for neutrino oscillations in uniform matter density so that the resulting oscillation probabilities are accurate for the complete matter potential versus baseline divided by neutrino energy plane. This extension also gives the *exact* oscillation probabilities in vacuum for all values of baseline divided by neutrino energy. The expansion parameter used is related to the ratio of the solar to the atmospheric Δm^2 scales but with a unique choice of the atmospheric Δm^2 such that certain first-order effects are taken into account in the zeroth-order Hamiltonian. Using a mixing matrix formulation, this framework has the exceptional feature that the neutrino oscillation probability in matter has the same structure as in vacuum, to all orders in the expansion parameter. It also contains all orders in the matter potential and $\sin\theta_{13}$. It facilitates immediate physical interpretation of the analytic results, and makes the expressions for the neutrino oscillation probabilities extremely compact and very accurate even at zeroth order in our perturbative expansion. The first and second order results are also given which improve the precision by approximately two or more orders of magnitude per perturbative order.

Analytic and compact perturbative expressions for neutrino oscillations in matter

Peter B. Denton

Fermilab Theory Seminar

July 21, 2016

Work done with S. Parke and H. Minakata.

1604.08167

github.com/PeterDenton/Nu-Pert

Fermilab

VANDERBILT UNIVERSITY

Takaaki Kajita
Super K

Arthur McDonald
SNO

"for the discovery of neutrino oscillations,
which shows that neutrinos have mass"

Neutrino oscillations in vacuum

Neutrino oscillations in matter are described by this Hamiltonian written in the flavor basis:

$$H = \frac{1}{2E} U \begin{pmatrix} m_1^2 & & \\ & m_2^2 & \\ & & m_3^2 \end{pmatrix} U^\dagger \,,$$

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where the unitary mixing matrix is parameterized by

$$U = \begin{pmatrix} 1 & & \\ & c_{23} & s_{23}e^{i\delta} \\ & -s_{23}e^{-i\delta} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & & s_{13} \\ & 1 & \\ -s_{13} & & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & \\ -s_{12} & c_{12} & \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_{12}c_{13} & c_{13}s_{12} & s_{13} \\ -c_{23}s_{12} - e^{i\delta}c_{12}s_{13}s_{23} & c_{12}c_{23} - e^{i\delta}s_{12}s_{13}s_{23} & e^{i\delta}c_{13}s_{23} \\ e^{-i\delta}s_{12}s_{23} - c_{12}c_{23}s_{13} & -c_{23}s_{12}s_{13} - e^{-i\delta}c_{12}s_{23} & c_{13}c_{23} \end{pmatrix} .$$

Neutrino oscillations in vacuum: disappearance

For example, it is easy to calculate the *exact* disappearance expression in vacuum:

$$P(\nu_\alpha \to \nu_\alpha) = 1 - 4 \sum_{i<j} |U_{\alpha i}|^2 |U_{\alpha j}|^2 \sin^2 \Delta_{ji} \, .$$

$$\Delta_{ij} = \frac{\Delta m^2_{ij} L}{4E}$$
$$\Delta m^2_{ij} = m^2_i - m^2_j$$

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For the electron case this expression is simple:

$$\begin{aligned}P(\nu_e \to \nu_e) = 1 & \\ & - 4c_{12}^2 s_{12}^2 c_{13}^4 \sin^2 \Delta_{21} \\ & - 4c_{12}^2 c_{13}^2 s_{13}^2 \sin^2 \Delta_{31} \\ & - 4s_{12}^2 c_{13}^2 s_{13}^2 \sin^2 \Delta_{32}\,.\end{aligned}$$

$$\Delta_{ij} = \frac{\Delta m_{ij}^2 L}{4E}$$
$$\Delta m_{ij}^2 = m_i^2 - m_j^2$$

Neutrino oscillations in vacuum: appearance

For the $\nu_\mu \to \nu_e$ appearance channel,

$$A_{\mu e} = A_{31} + A_{21} e^{i(\delta+\Delta_{32})} ,$$

where

$$A_{31} = 2|U^*_{\mu 3}|U_{e3} \sin \Delta_{31} = 2 s_{23} c_{13} s_{13} \sin \Delta_{31} ,$$
$$A_{21} = 2 U^*_{\mu 2} U_{e2} \sin \Delta_{21} \approx 2 \left[c_{12} c_{23} s_{12} c_{13} + \mathcal{O}(s_{13}) \right] \sin \Delta_{21} .$$

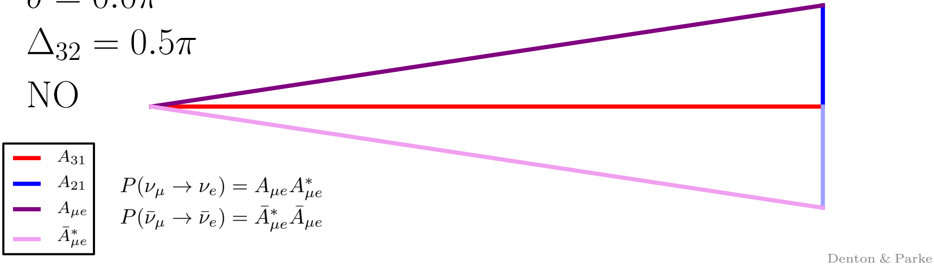
Neutrino oscillations in vacuum: appearance

Neutrino oscillations in vacuum: appearance

$\delta = 0.0\pi$

$\Delta_{32} = 0.5\pi$

NO



$$P(\nu_\mu \to \nu_e) = A_{\mu e} A^*_{\mu e}$$

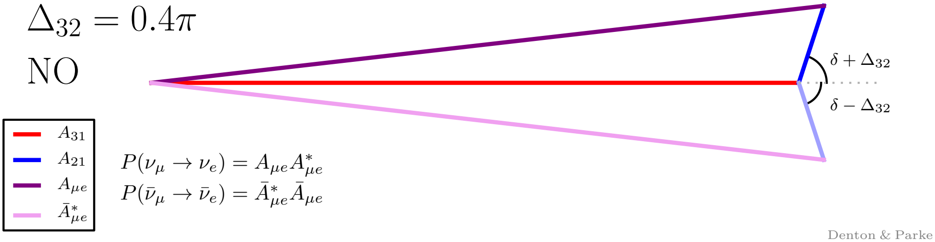
$$P(\bar{\nu}_\mu \to \bar{\nu}_e) = \bar{A}^*_{\mu e} \bar{A}_{\mu e}$$

Neutrino oscillations in vacuum: appearance

$\delta = 0.0\pi$

$\Delta_{32} = 0.4\pi$

NO



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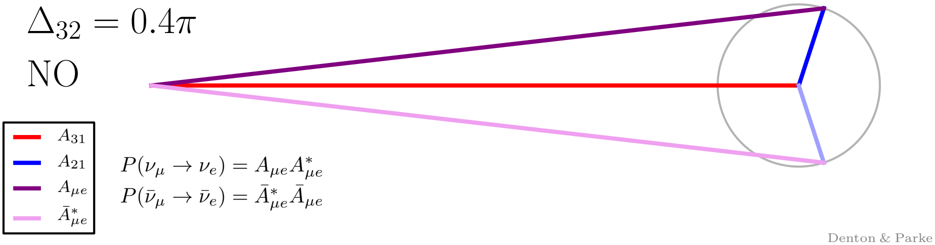

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Neutrino properties

Current status:

- $\sin^2\theta_{13} \sim 0.02$.
- $\sin^2\theta_{12} \sim 0.3$.
- $\sin^2\theta_{23} \sim 0.5$.
- $\delta \sim$???

- $\Delta m^2_{21} = 7.6 \times 10^{-5}$ eV^2.
- $|\Delta m^2_{31}| \approx |\Delta m^2_{32}| \sim 2.4 \times 10^{-3}$ eV^2.
- $\frac{|\Delta m^2_{31}|}{\Delta m^2_{21}} \sim 30$.

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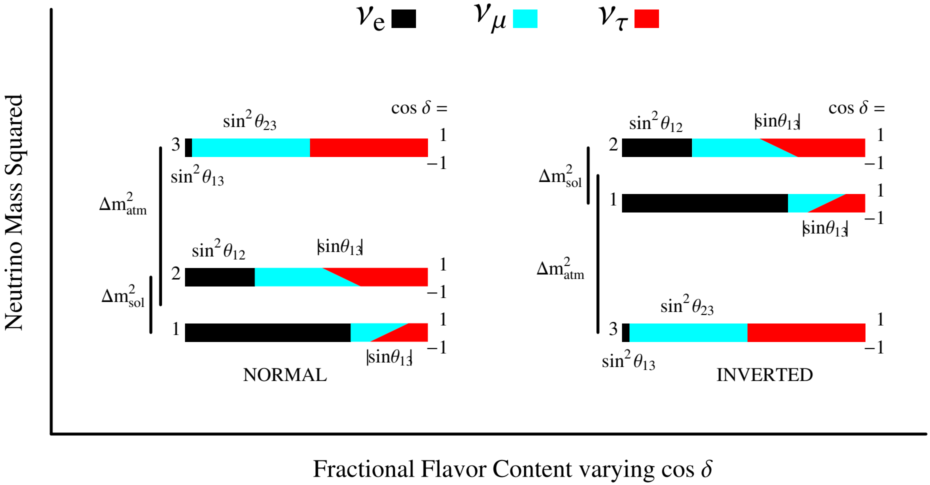
Unknowns:

- δ: CP phase.
- $\sin^2\theta_{23} < 0.5$ or $\sin^2\theta_{23} > 0.5$: octant (dominant flavor ν_3).
- $\Delta m^2_{31} > 0$ or $\Delta m^2_{31} < 0$: mass ordering.
- m_1^2: absolute mass scale.
- α_1, α_2: Majorana phases.
- Anomalies: Short baseline, reactor, gallium, ...
- New physics: NSI, magnetic moment, decay, steriles, ...

Neutrino properties



O. Mena, S. Parke, hep-ph/0312131

Matter effects

Oscillation probabilities are different in matter.

L. Wolfenstein, PRD 17 (1978)

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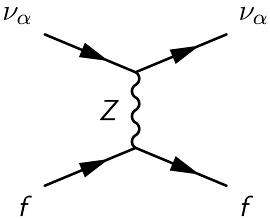

$$V_{\mathrm{NC}} = \mp\sqrt{2}G_F\frac{1}{2}N_n$$

$N_{n(e)}$ is the neutron (electron) number density,
upper (lower) sign is for (anti) neutrinos.

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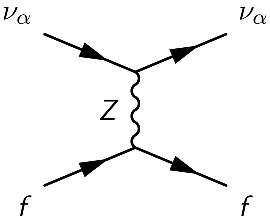
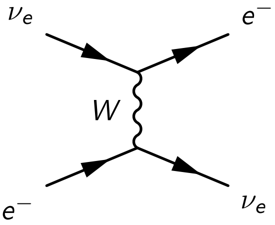


$$V_{\rm NC} = \mp\sqrt{2}G_F\frac{1}{2}N_n$$



$$V_{\rm CC} = \pm\sqrt{2}G_F N_e$$

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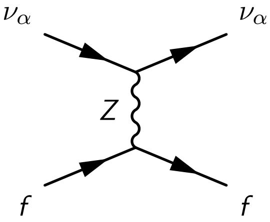

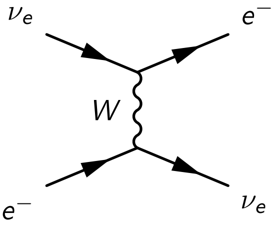

$$V_{\mathrm{NC}} = \mp\sqrt{2}G_F\frac{1}{2}N_n \qquad V_{\mathrm{CC}} = \pm\sqrt{2}G_F N_e$$

Resolved the solar neutrino problem (MSW effect).

S. Mikheyev, A. Smirnov, Sov. J. Nucl. Phys. 42 (1985)

$N_{n(e)}$ is the neutron (electron) number density,
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Identity matrix

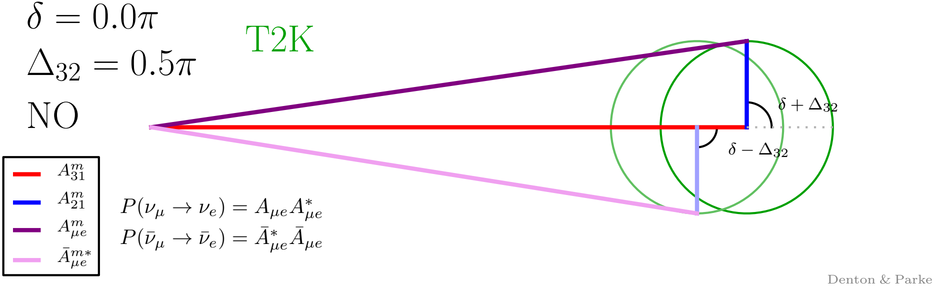
Oscillation probabilities are invariant under

$$H \to H + x\mathbb{1}\,,$$

for some $x \in \mathbb{R}$.

So oscillation experiments cannot tell us about the absolute mass scale or $V_{\rm NC}$.

Matter effect at long baseline experiments



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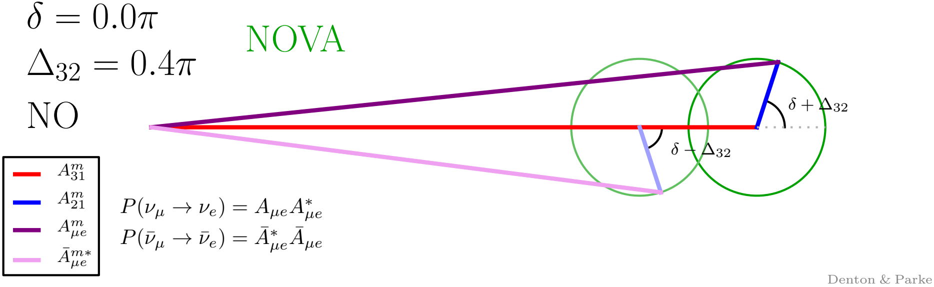

Matter effect at long baseline experiments

$\delta = 0.0\pi$

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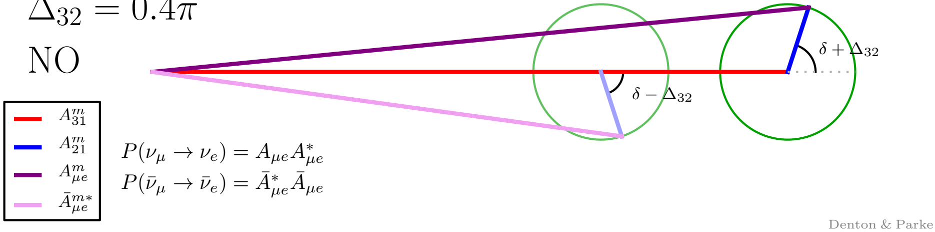
NO

DUNE



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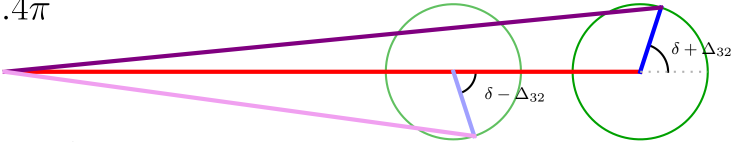

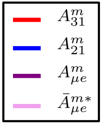

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Denton & Parke

NO: $P > \bar{P} \,\forall\, \delta$

IO: $P < \bar{P} \,\forall\, \delta$

Neutrino oscillations in matter

The electron *flavor* interacts differently, modifying the Hamiltonian.

$$H^m = \frac{1}{2E}\left[U\begin{pmatrix}0 & & \\ & \Delta m^2_{21} & \\ & & \Delta m^2_{31}\end{pmatrix}U^\dagger + \begin{pmatrix}a & & \\ & & \\ & & \end{pmatrix}\right].$$

$$\begin{aligned}U &\equiv U_{23}(\theta_{23},\delta)U_{13}(\theta_{13})U_{12}(\theta_{12}),\\ a &= 2EV_{\mathrm{CC}} = 2\sqrt{2}G_F N_e E,\\ \Delta_{ij} &= \frac{\Delta m^2_{ij}L}{4E}.\end{aligned}$$

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We could write

$$H^m = \frac{1}{2E}U^m\begin{pmatrix}\lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3\end{pmatrix}U^{m\dagger},$$

where the λ_i are the masses squared in matter,
and $\theta^m_{12}, \theta^m_{13}, \theta^m_{23}$, and δ^m and the angles and phase in matter.

$$\begin{aligned} U &\equiv U_{23}(\theta_{23},\delta)U_{13}(\theta_{13})U_{12}(\theta_{12}),\\ a &= 2EV_{\rm CC} = 2\sqrt{2}G_F N_e E,\\ \Delta_{ij} &= \frac{\Delta m^2_{ij}L}{4E}.\end{aligned}$$

Exact neutrino oscillations in matter: mixing angles

This has been done.

H. Zaglauer, K. Schwarzer, Z. Phys. C Particles and Fields, 40 (1988)

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$$s_{12}^{m2} = \frac{-\left(\lambda_2^2 - \alpha\lambda_2 + \beta\right)\Delta\lambda_{31}}{\left(\lambda_1^2 - \alpha\lambda_1 + \beta\right)\Delta\lambda_{32} - \left(\lambda_2^2 - \alpha\lambda_2 + \beta\right)\Delta\lambda_{31}}$$

$$s_{13}^{m2} = \frac{\lambda_3^2 - \alpha\lambda_3 + \beta}{\Delta\lambda_{31}\Delta\lambda_{32}}$$

$$s_{23}^{m2} = \frac{s_{23}^2 E^2 + c_{23}^2 F^2 + 2c_{23}s_{23}c_\delta EF}{E^2 + F^2}$$

$$e^{-i\delta^m} = \frac{c_{23}^2 s_{23}^2\left(e^{-i\delta}E^2 - e^{i\delta}F^2\right) + \left(c_{23}^2 - s_{23}^2\right)EF}{\sqrt{\left(s_{23}^2E^2 + c_{23}^2F^2 + 2EFc_{23}s_{23}c_\delta\right)\left(c_{23}^2E^2 + s_{23}^2F^2 - 2EFc_{23}s_{23}c_\delta\right)}}$$

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$$\alpha = c_{13}^2\Delta m_{31}^2 + \left(c_{12}^2 c_{13}^2 + s_{13}^2\right)\Delta m_{21}^2\,,\ \beta = c_{12}^2 c_{13}^2\Delta m_{21}^2\Delta m_{31}^2$$

$$E = c_{13}s_{13}\left[\left(\lambda_3 - \Delta m_{21}^2\right)\Delta m_{31}^2 - s_{12}^2\left(\lambda_3 - \Delta m_{31}^2\right)\Delta m_{21}^2\right]$$

$$F = c_{12}s_{12}c_{13}\left(\lambda_3 - \Delta m_{31}^2\right)\Delta m_{21}^2$$

Exact neutrino oscillations in matter: eigenvalues

$$\lambda_1 = \frac{A}{3} - \frac{1}{3}\sqrt{A^2 - 3B}\,S - \frac{\sqrt{3}}{3}\sqrt{A^2 - 3B}\sqrt{1 - S^2}$$

$$\lambda_2 = \frac{A}{3} - \frac{1}{3}\sqrt{A^2 - 3B}\,S + \frac{\sqrt{3}}{3}\sqrt{A^2 - 3B}\sqrt{1 - S^2}$$

$$\lambda_3 = \frac{A}{3} + \frac{2}{3}\sqrt{A^2 - 3B}\,S$$

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$$A = \Delta m^2_{21} + \Delta m^2_{31} + a$$
$$B = \Delta m^2_{21}\Delta m^2_{31} + a\left[c^2_{13}\Delta m^2_{31} + (c^2_{12}c^2_{13} + s^2_{13})\Delta m^2_{21}\right]$$

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$$S = \cos\left\{\frac{1}{3}\cos^{-1}\left[\frac{2A^3 - 9AB + 27C}{2\left(A^2-3B\right)^{3/2}}\right]\right\}$$

$$C = a\Delta m^2_{21}\Delta m^2_{31}c^2_{12}c^2_{13}$$

Alternative solutions

- Numerical methods:
  - Good for experiments.
  - Important understanding can be missed (magic baseline).

P. Huber, W. Winter, hep-ph/0301257

A. Smirnov, hep-ph/0610198

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P. Huber, W. Winter, hep-ph/0301257

A. Smirnov, hep-ph/0610198

- Perturbative expansion:
  - Small matter potential: $a/\Delta m^2$.

Y. Li, Y. Wang, Z-z. Xing, 1605.00900

  - s_{13}, s_{13}^2.

A. Cervera, et. al., hep-ph/0002108

H. Minakata, 0910.5545

K. Asano, H. Minakata, 1103.4387

  - $\Delta m^2_{21}/\Delta m^2_{31} \sim 0.03$.

J. Arafune, J. Sato, hep-ph/9607437

A. Cervera, et. al., hep-ph/0002108

M. Freund, hep-ph/0103300

E. Akhmedov, et. al., hep-ph/0402175

H. Minakata, S. Parke, 1505.01826

PBD, H. Minakata, S. Parke, 1604.08167

Alternative solutions: example

$$P_0 = \sin^2\theta_{23}\frac{\sin^2 2\theta_{13}}{\hat{C}^2}\sin^2(\hat{\Delta}\hat{C}), \tag{36a}$$

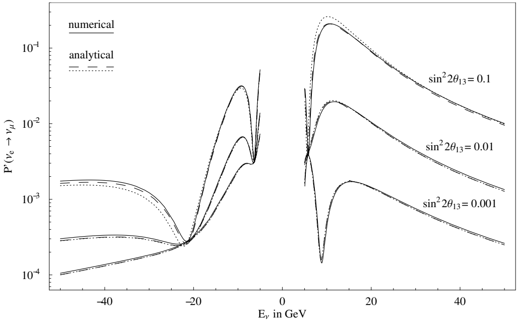
$$P_{\sin\delta} = \frac{1}{2}\alpha\frac{\sin\delta\cos\theta_{13}\sin 2\theta_{12}\sin 2\theta_{13}\sin 2\theta_{23}}{\hat{A}\hat{C}\cos\theta_{13}^2}\sin(\hat{C}\hat{\Delta})$$
$$\times\{\cos(\hat{C}\hat{\Delta}) - \cos((1+\hat{A})\hat{\Delta})\}, \tag{36b}$$

$$P_{\cos\delta} = \frac{1}{2}\alpha\frac{\cos\delta\cos\theta_{13}\sin 2\theta_{12}\sin 2\theta_{13}\sin 2\theta_{23}}{\hat{A}\hat{C}\cos\theta_{13}^2}\sin(\hat{C}\hat{\Delta})$$
$$\times\{\sin((1+\hat{A})\hat{\Delta}) \mp \sin(\hat{C}\hat{\Delta})\}, \tag{36c}$$

$$P_1 = -\alpha\frac{1-\hat{A}\cos 2\theta_{13}}{\hat{C}^3}\sin^2\theta_{12}\sin^2 2\theta_{13}\sin^2\theta_{23}\hat{\Delta}$$
$$\times\sin(2\hat{\Delta}\hat{C}) + \alpha\frac{2\hat{A}(-\hat{A}+\cos 2\theta_{13})}{\hat{C}^4}$$
$$\times\sin^2\theta_{12}\sin^2 2\theta_{13}\sin^2\theta_{23}\sin^2(\hat{\Delta}\hat{C}), \tag{36d}$$

$$P_2 = \alpha\frac{\mp 1 + \hat{C} \pm \hat{A}\cos 2\theta_{13}}{2\hat{C}^2\hat{A}\cos^2\theta_{13}}\cos\theta_{13}\sin 2\theta_{12}\sin 2\theta_{13}$$
$$\times\sin 2\theta_{23}\sin^2(\hat{\Delta}\hat{C}), \tag{36e}$$

$$P_3 = \alpha^2\frac{2\hat{C}\cos^2\theta_{23}\sin^2 2\theta_{12}}{\hat{A}^2\cos^2\theta_{13}(\mp\hat{A}+\hat{C}\pm\cos 2\theta_{13})}$$
$$\times\sin^2\left(\frac{1}{2}(1+\hat{A}\mp\hat{C})\hat{\Delta}\right). \tag{36f}$$



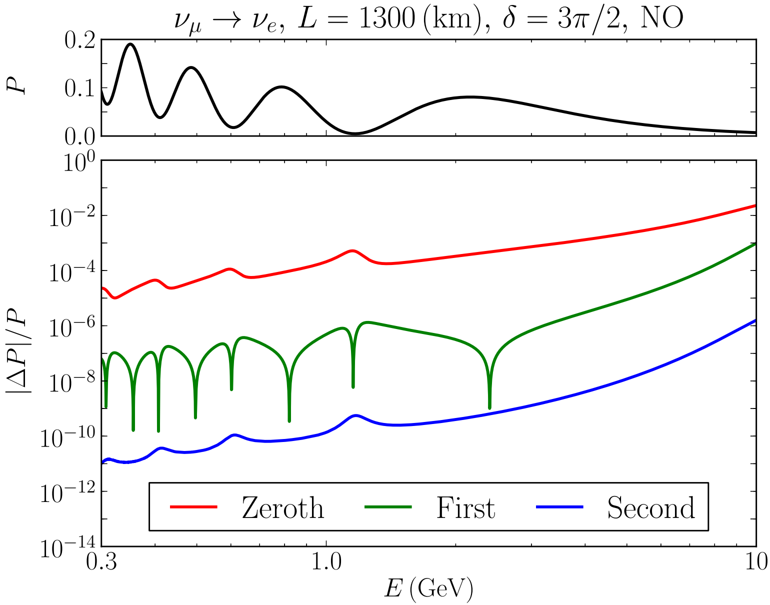
M. Freund, hep-ph/0103300

A perturbative description of neutrino oscillations in matter

Several goals:

1. Use $\epsilon \simeq \Delta m^2_{21}/\Delta m^2_{31} = 0.03$ as an expansion parameter.
2. Minimize the number of additional 'matter' angles/phases (2).
3. Extremely accurate for all channels, L/E, orderings, CP phases, ...
4. Exact in the vacuum limit.
5. Reproduce the known CPV term in matter.

Precision order-by-order



Our methodology

- Start with $\epsilon = \frac{\Delta m^2_{21}}{\Delta m^2_{ee}} = 0.03$.

$$\Delta m^2_{ee} = \Delta m^2_{31} - s^2_{12}\Delta m^2_{21}$$

H. Nunokawa, S. Parke, R. Zukanovich Funchal, hep-ph/0503283

Our methodology

- Start with $\epsilon = \frac{\Delta m^2_{21}}{\Delta m^2_{ee}} = 0.03$.
- Perform one fixed and two variable rotations: (θ_{23}, δ), ϕ, ψ.

$$\Delta m^2_{ee} = \Delta m^2_{31} - s^2_{12} \Delta m^2_{21}$$

H. Nunokawa, S. Parke, R. Zukanovich Funchal, hep-ph/0503283

Our methodology

- Start with $\epsilon = \frac{\Delta m^2_{21}}{\Delta m^2_{ee}} = 0.03$.
- Perform one fixed and two variable rotations: (θ_{23}, δ), ϕ, ψ.
- Write the probabilities with simple L/E dependence:

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - \sum_{i<j} \Re\left[U_{\alpha i} U^*_{\beta i} U^*_{\alpha j} U_{\beta j}\right] \sin^2 \Delta_{ij}$$
$$+ 8\Im\left[U_{\alpha 1} U^*_{\beta 2} U^*_{\alpha 2} U_{\beta 1}\right] \sin \Delta_{32} \sin \Delta_{31} \sin \Delta_{21}$$

C. Jarlskog: PRL 55 (1985)

Nonvanishing Wronskian $\Rightarrow$ fewest number of L/E functions.

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Clear that the CPV term is $\mathcal{O}[(L/E)^3]$ not $\mathcal{O}[(L/E)^1]$.

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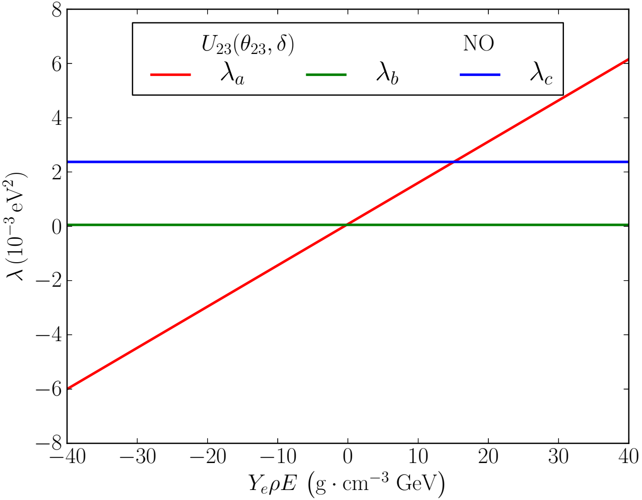
Eigenvalues in matter: two rotations are needed



$$\lambda_a = a + (s_{13}^2 + \epsilon s_{12}^2)\Delta m_{ee}^2\,, \quad \lambda_b = \epsilon c_{12}^2 \Delta m_{ee}^2\,, \quad \lambda_c = (c_{13}^2 + \epsilon s_{12}^2)\Delta m_{ee}^2\,.$$

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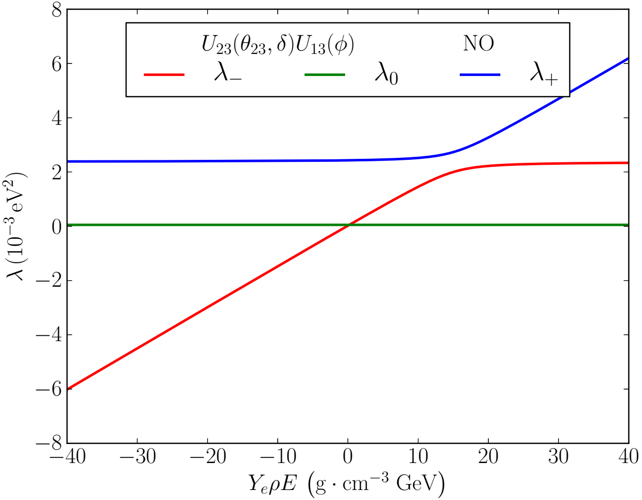

$$\lambda_\mp = \frac{1}{2}\left[(\lambda_a + \lambda_c) \mp \mathrm{sign}(\Delta m^2_{ee})\sqrt{(\lambda_c - \lambda_a)^2 + 4(s_{13}c_{13}\Delta m^2_{ee})^2}\right] ,$$

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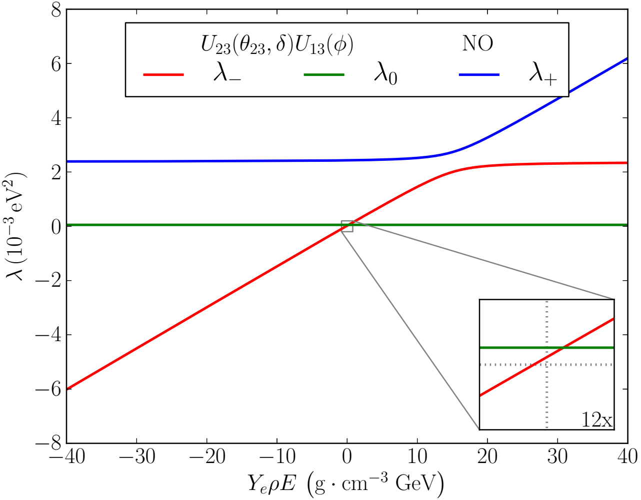

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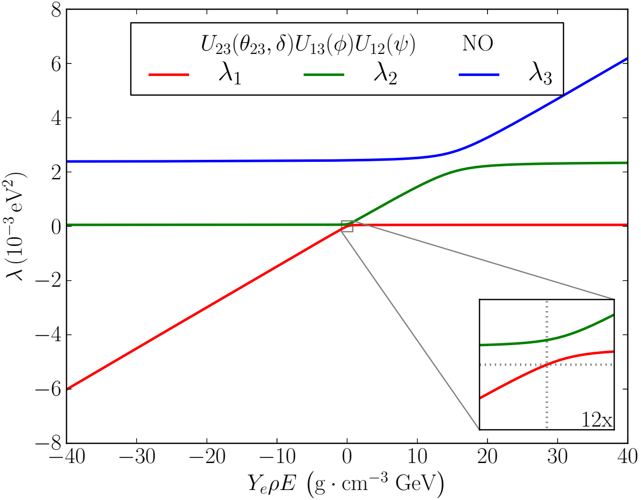
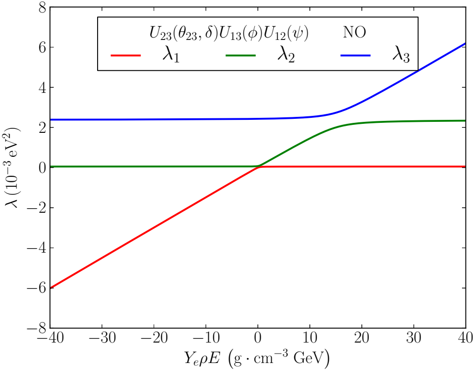
Eigenvalues in matter: two rotations are needed

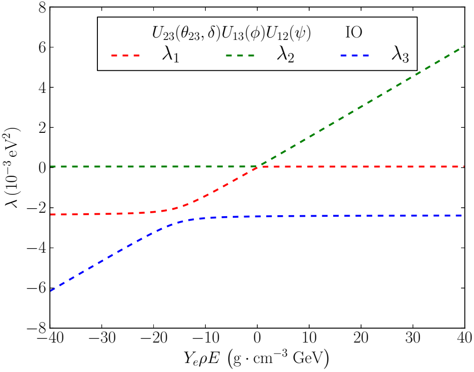


$$\lambda_{1,2} = \frac{1}{2}\left[(\lambda_0 + \lambda_-) \mp \sqrt{(\lambda_0 - \lambda_-)^2 + 4(\epsilon c_{(\phi-\theta_{13})} c_{12} s_{12} \Delta m^2_{ee})^2}\right],$$

$$\lambda_3 = \lambda_+ .$$

Eigenvalues in matter: mass ordering





NO

$$\lambda_1 < \lambda_2 < \lambda_3$$

IO

$$\lambda_3 < \lambda_1 < \lambda_2$$

$1+2$ rotations

1. Perform a constant $U_{23}(\theta_{23}, \delta)$ rotation.
   - U_{23} commutes with the matter potential.
   - Resultant Hamiltonian is real.
   - 'Expansion parameter' is $c_{13}s_{13} = 0.15$ at this point.

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2. Diagonalize the diagonal and $\mathcal{O}(\epsilon^0)$ off-diagonal terms with $U_{13}(\phi)$.
   - $\phi(a = 0) = \theta_{13}$.
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H. Minakata, S. Parke, 1505.01826

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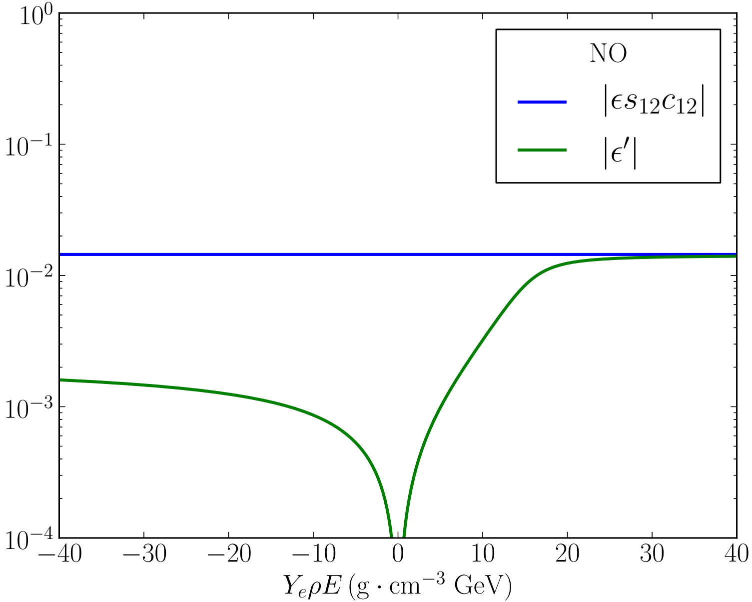
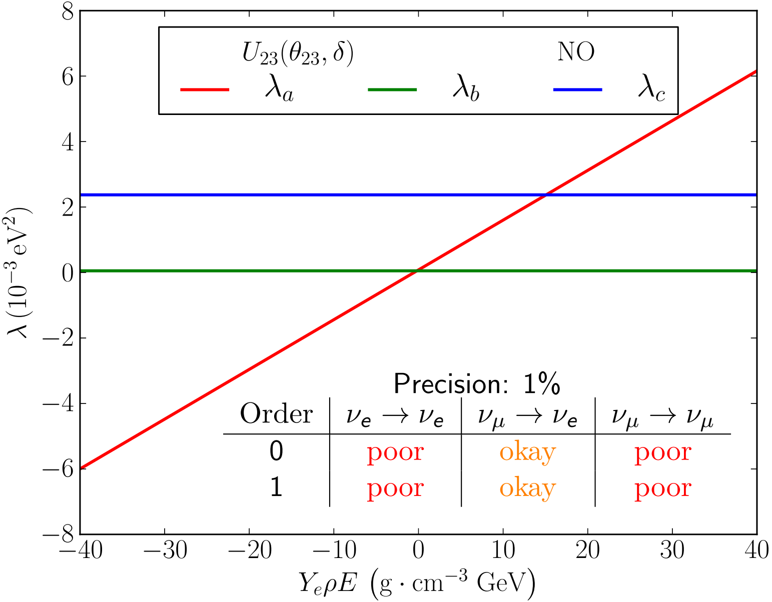
3. Diagonalize the terms non-zero in vacuum with $U_{12}(\psi)$.
   - $\psi(a=0)=\theta_{12}$.
   - Expansion parameter is now $\epsilon' = c_{12}s_{12}s_{(\phi-\theta_{13})}\frac{\Delta m^2_{21}}{\Delta m^2_{ee}} < 0.015$.
   - $\epsilon'(a=0)=0$.

Expansion parameter

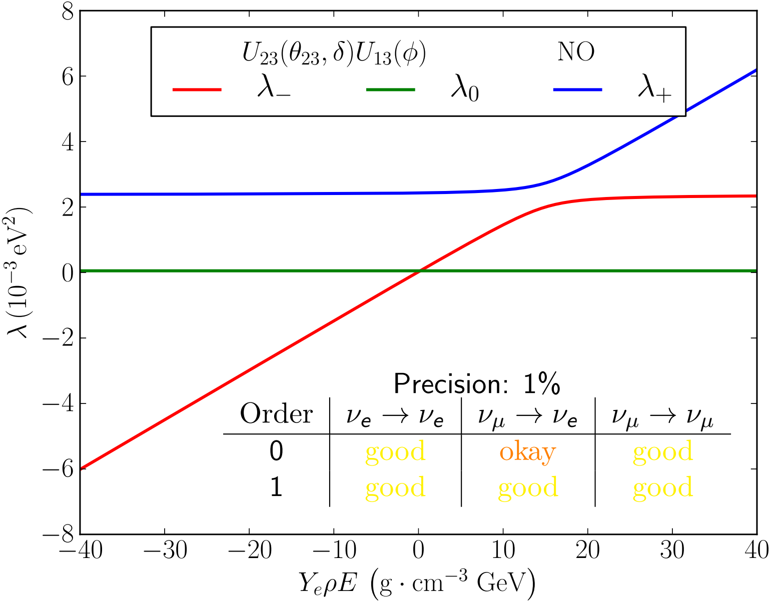


Eigenvalues in matter: two rotations are needed



Order	$\nu_e \to \nu_e$	$\nu_\mu \to \nu_e$	$\nu_\mu \to \nu_\mu$
0	poor	okay	poor
1	poor	okay	poor

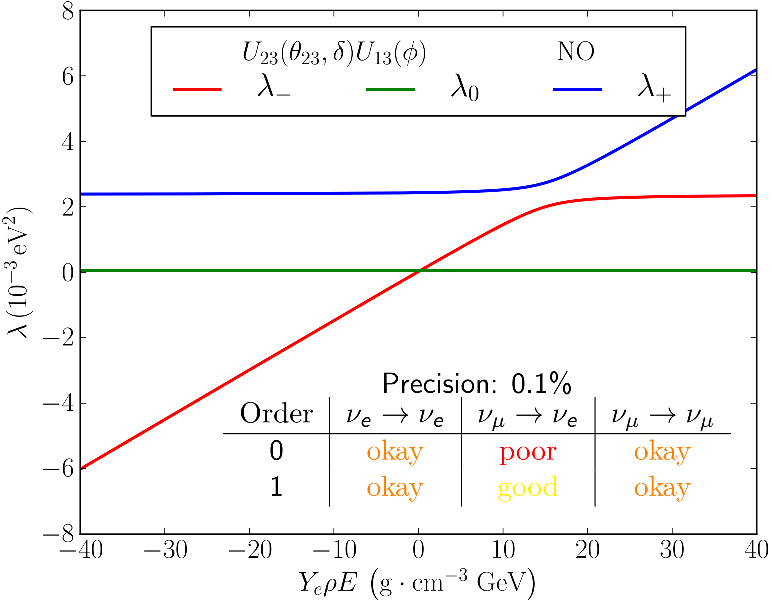
Eigenvalues in matter: two rotations are needed



Precision: 1%

Order	$\nu_e \to \nu_e$	$\nu_\mu \to \nu_e$	$\nu_\mu \to \nu_\mu$
0	good	okay	good
1	good	good	good

Eigenvalues in matter: two rotations are needed



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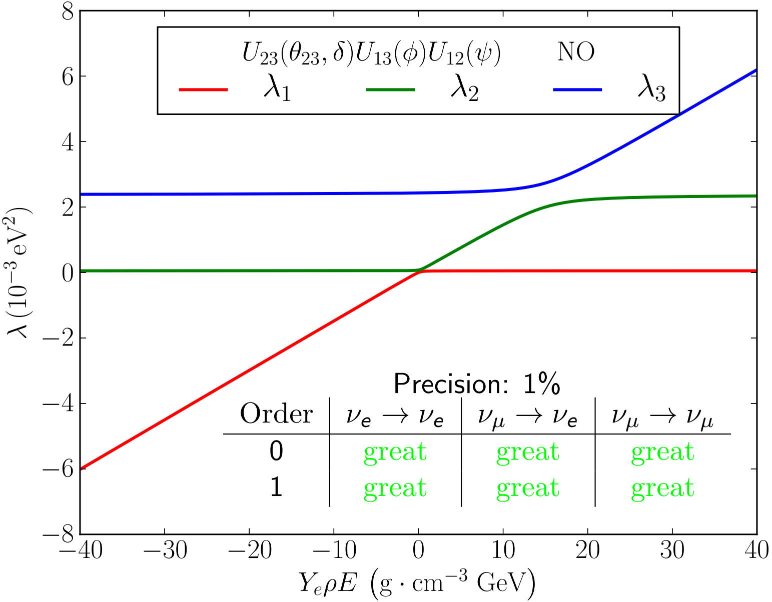
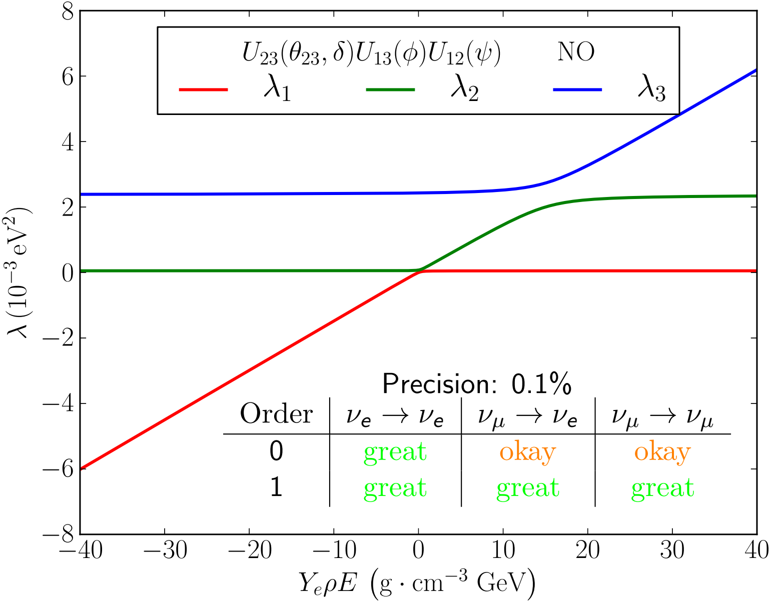


Precision: 1%

Order	$\nu_e \to \nu_e$	$\nu_\mu \to \nu_e$	$\nu_\mu \to \nu_\mu$
0	great	great	great
1	great	great	great

Eigenvalues in matter: two rotations are needed



Precision: 0.1%

Order	$\nu_e \to \nu_e$	$\nu_\mu \to \nu_e$	$\nu_\mu \to \nu_\mu$
0	great	okay	okay
1	great	great	great

Simplicity at zeroth order

Simple L/E dependence:

$$P = \delta^{\alpha\beta} + 4C_{21}^{\alpha\beta}\sin^2\Delta_{21} + 4C_{31}^{\alpha\beta}\sin^2\Delta_{31} + 4C_{32}^{\alpha\beta}\sin^2\Delta_{32}$$

$$+8D^{\alpha\beta}\sin\Delta_{21}\sin\Delta_{31}\sin\Delta_{32}\,.$$

$$\Delta_{ij} = \frac{\Delta m_{ij}^2 L}{4E}$$

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$$+8D^{\alpha\beta}\sin\Delta_{21}\sin\Delta_{31}\sin\Delta_{32}\,.$$

Zeroth order is equivalent to the vacuum probability with

$$\theta_{13}\to\phi\,,\qquad \theta_{12}\to\psi\,,$$

$$\Delta m^2_{ij}\to\Delta\lambda_{ij}\,.$$

$$\Delta_{ij} = \frac{\Delta m^2_{ij} L}{4E}$$

Eigenvalues

Tilde basis after $U_{23}(\theta_{23}, \delta)$:

$$\begin{aligned}
\lambda_a &= a + (s_{13}^2 + \epsilon s_{12}^2)\Delta m_{ee}^2\,, \\
\lambda_b &= \epsilon c_{12}^2 \Delta m_{ee}^2\,, \\
\lambda_c &= (c_{13}^2 + \epsilon s_{12}^2)\Delta m_{ee}^2\,.
\end{aligned}$$

Eigenvalues

Tilde basis after $U_{23}(\theta_{23},\delta)$:

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Hat basis after $U_{13}(\phi)$:

$$\lambda_{\mp} = \frac{1}{2}\left[(\lambda_a + \lambda_c) \mp \operatorname{sign}(\Delta m_{ee}^2)\sqrt{(\lambda_c - \lambda_a)^2 + 4(s_{13}c_{13}\Delta m_{ee}^2)^2}\right]\,,$$
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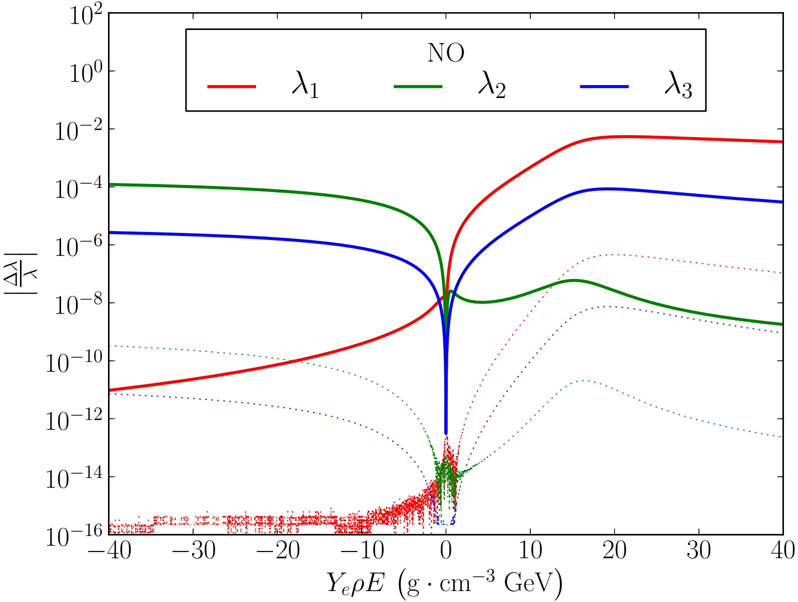
Check basis after $U_{12}(\psi)$:

$$\lambda_{1,2} = \frac{1}{2}\left[(\lambda_0 + \lambda_-) \mp \sqrt{(\lambda_0 - \lambda_-)^2 + 4(\epsilon c_{(\phi-\theta_{13})} c_{12} s_{12}\Delta m_{ee}^2)^2}\right]\,,$$
$$\lambda_3 = \lambda_+\,.$$

Eigenvalues: precision



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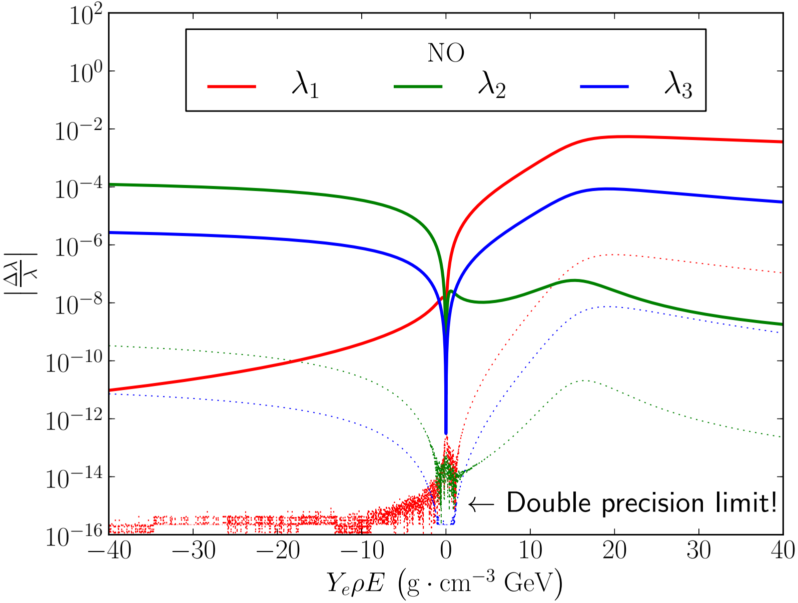

Angles in matter

The $\theta_{13} \to \phi$ angle in matter:

$$c_{2\phi} = \frac{\lambda_c - \lambda_a}{\lambda_+ - \lambda_-}\,, \qquad s_{2\phi} = s_{2\theta_{13}} \frac{\Delta m^2_{ee}}{\lambda_+ - \lambda_-}\,.$$

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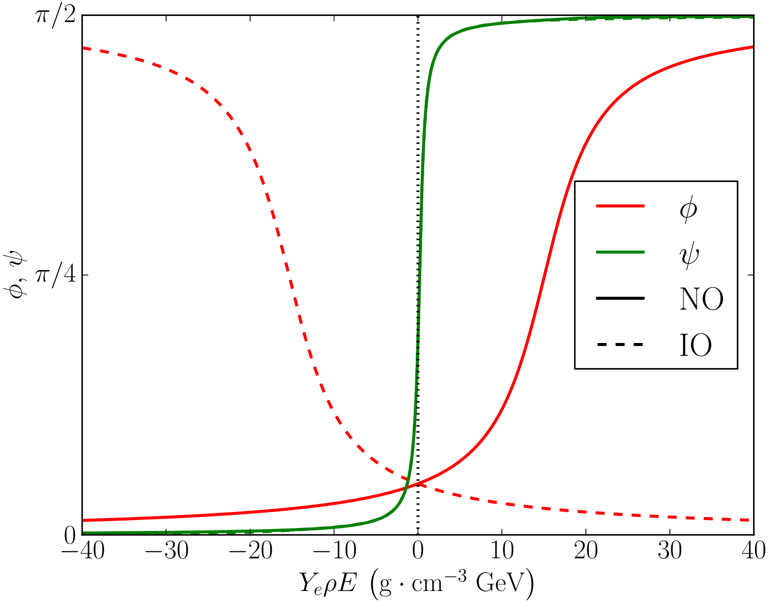
The $\theta_{12} \to \psi$ angle matter:

$$c_{2\psi} = \frac{\lambda_0 - \lambda_-}{\lambda_2 - \lambda_1}\,, \qquad s_{2\psi} = s_{2\theta_{12}} \epsilon c_{(\phi-\theta_{13})} \frac{\Delta m^2_{ee}}{\lambda_2 - \lambda_1}\,.$$

The two angles in matter



Hamiltonians

After a constant (θ_{23}, δ) rotation, $2E\tilde{H} =$

$$\begin{pmatrix} \lambda_a & & s_{13}c_{13}\Delta m^2_{ee} \\ & \lambda_b & \\ s_{13}c_{13}\Delta m^2_{ee} & & \lambda_c \end{pmatrix} + \epsilon s_{12}c_{12}\Delta m^2_{ee} \begin{pmatrix} & c_{13} & \\ c_{13} & & -s_{13} \\ & -s_{13} & \end{pmatrix} .$$

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After a $U_{13}(\phi)$ rotation, $2E\hat{H} =$

$$\begin{pmatrix} \lambda_- & & \\ & \lambda_0 & \\ & & \lambda_+ \end{pmatrix} + \epsilon c_{12}s_{12}\Delta m^2_{ee} \begin{pmatrix} & c_{(\phi-\theta_{13})} & \\ c_{(\phi-\theta_{13})} & & s_{(\phi-\theta_{13})} \\ & s_{(\phi-\theta_{13})} & \end{pmatrix} .$$

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After a $U_{12}(\psi)$ rotation, $2E\check{H} =$

$$\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} + \epsilon {\color{red}s_{(\phi-\theta_{13})}} s_{12}c_{12}\Delta m^2_{ee} \begin{pmatrix} & & -s_\psi \\ & & c_\psi \\ -s_\psi & c_\psi & \end{pmatrix} .$$

$\lambda_{1,2} - \psi$ interchange

From the shape of $U_{12}(\psi)$, it is clear that the probabilities are invariant under a simultaneous interchange of

$$\lambda_1 \leftrightarrow \lambda_2 \,, \qquad \text{and} \qquad \psi \to \psi \pm \frac{\pi}{2} \,.$$

Since only even powers of ψ trig functions $(c_\psi^2, s_\psi^2, c_\psi s_\psi, c_{2\psi}, s_{2\psi})$ appear in the probabilities, the sign degeneracy is irrelevant. More usefully, we can write that the probabilities are invariant under the simultaneous interchange of

$$\lambda_1 \leftrightarrow \lambda_2 \,, \qquad c_\psi^2 \leftrightarrow s_\psi^2 \,, \qquad \text{and} \qquad c_\psi s_\psi \to -c_\psi s_\psi \,.$$

This interchange constrains C_{21},
and C_{32} is then easily calculated from C_{31}.

Perturbative expansion

Hamiltonian: $\check{H} = \check{H}_0 + \check{H}_1$.

$$\check{H}_0 = \frac{1}{2E}\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}\,, \quad \check{H}_1 = \epsilon' \frac{\Delta m^2_{ee}}{2E}\begin{pmatrix} & & -s_\psi \\ & & c_\psi \\ -s_\psi & c_\psi & \end{pmatrix}\,.$$

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Eigenvalues: $\lambda_i^{\rm ex} = \lambda_i + \lambda_i^{(1)} + \lambda_i^{(2)} + \cdots$.

$$\lambda_i^{(1)} = 2E(\check{H}_1)_{ii} = 0\,,$$

$$\lambda_i^{(2)} = \sum_{k\neq i} \frac{[2E(\check{H}_1)_{ik}]^2}{\Delta\lambda_{ik}}\,.$$

Perturbative expansion: eigenvectors

Use vacuum expressions with $U \to V$ where

$$V = U^m W\,,$$

U^m is U with $\theta_{13} \to \phi$ and $\theta_{12} \to \psi$,

$W = W_0 + W_1 + W_2 + \cdots$.

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$$W_1 = \epsilon' \Delta m^2_{ee} \begin{pmatrix} & & -\frac{s_\psi}{\Delta\lambda_{31}} \\ & & \frac{c_\psi}{\Delta\lambda_{32}} \\ \frac{s_\psi}{\Delta\lambda_{31}} & -\frac{c_\psi}{\Delta\lambda_{32}} & \end{pmatrix}\,,$$

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$$W_2 = -\epsilon'^2 \frac{(\Delta m^2_{ee})^2}{2} \begin{pmatrix} \frac{s_\psi^2}{(\Delta\lambda_{31})^2} & -\frac{s_{2\psi}}{\Delta\lambda_{32}\Delta\lambda_{21}} & \\ \frac{s_{2\psi}}{\Delta\lambda_{31}\Delta\lambda_{21}} & \frac{c_\psi^2}{(\Delta\lambda_{32})^2} & \\ & & \left[\frac{c_\psi^2}{(\Delta\lambda_{32})^2} + \frac{s_\psi^2}{(\Delta\lambda_{31})^2}\right] \end{pmatrix} \,.$$

Zeroth order coefficients

$\nu_\alpha \to \nu_\beta$	$(C_{21}^{\alpha\beta})^{(0)}$
$\nu_e \to \nu_e$	$-c_\phi^4 s_\psi^2 c_\psi^2$
$\nu_\mu \to \nu_e$	$c_\phi^2 s_\psi^2 c_\psi^2 (c_{23}^2 - s_\phi^2 s_{23}^2) + c_{2\psi} J_r^m \cos\delta$
$\nu_\mu \to \nu_\mu$	$-(c_{23}^2 c_\psi^2 + s_{23}^2 s_\phi^2 s_\psi^2)(c_{23}^2 s_\psi^2 + s_{23}^2 s_\phi^2 c_\psi^2)$ $-2(c_{23}^2 - s_\phi^2 s_{23}^2) c_{2\psi} J_{rr}^m \cos\delta + (2 J_{rr}^m \cos\delta)^2$

$\nu_\alpha \to \nu_\beta$	$(C_{31}^{\alpha\beta})^{(0)}$	$(D^{\alpha\beta})^{(0)}$
$\nu_e \to \nu_e$	$-c_\phi^2 s_\phi^2 c_\psi^2$	0
$\nu_\mu \to \nu_e$	$s_\phi^2 c_\phi^2 c_\psi^2 s_{23}^2 + J_r^m \cos\delta$	$-J_r^m \sin\delta$
$\nu_\mu \to \nu_\mu$	$-c_\phi^2 s_{23}^2 (c_{23}^2 s_\psi^2 + s_{23}^2 s_\phi^2 c_\psi^2)$ $-2 s_{23}^2 J_r^m \cos\delta$	0

$$J_r^m \equiv s_\psi c_\psi s_\phi c_\phi^2 s_{23} c_{23}$$
$$J_{rr}^m \equiv J_r^m / c_\phi^2$$

General form of the first order coefficients

Can reduce 8 expressions down to 3:

$$(C_{21}^{\alpha\beta})^{(1)} = \epsilon' \Delta m_{ee}^2 \left(\frac{F_1^{\alpha\beta}}{\Delta\lambda_{31}} + \frac{F_2^{\alpha\beta}}{\Delta\lambda_{32}} \right) ,$$

$$(C_{31}^{\alpha\beta})^{(1)} = \epsilon' \Delta m_{ee}^2 \left(\frac{F_1^{\alpha\beta} + G_1^{\alpha\beta}}{\Delta\lambda_{31}} - \frac{F_2^{\alpha\beta}}{\Delta\lambda_{32}} \right) ,$$

$$(C_{32}^{\alpha\beta})^{(1)} = \epsilon' \Delta m_{ee}^2 \left(-\frac{F_1^{\alpha\beta}}{\Delta\lambda_{31}} + \frac{F_2^{\alpha\beta} + G_2^{\alpha\beta}}{\Delta\lambda_{32}} \right) ,$$

$$(D^{\alpha\beta})^{(1)} = \epsilon' \Delta m_{ee}^2 \left(\frac{K_1^{\alpha\beta}}{\Delta\lambda_{31}} - \frac{K_2^{\alpha\beta}}{\Delta\lambda_{32}} \right) .$$

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$$(C_{31}^{\alpha\beta})^{(1)} = \epsilon' \Delta m_{ee}^2 \left(\frac{F_1^{\alpha\beta} + G_1^{\alpha\beta}}{\Delta\lambda_{31}} - \frac{F_2^{\alpha\beta}}{\Delta\lambda_{32}} \right) ,$$

$$(C_{32}^{\alpha\beta})^{(1)} = \epsilon' \Delta m_{ee}^2 \left(-\frac{F_1^{\alpha\beta}}{\Delta\lambda_{31}} + \frac{F_2^{\alpha\beta} + G_2^{\alpha\beta}}{\Delta\lambda_{32}} \right) ,$$

$$(D^{\alpha\beta})^{(1)} = \epsilon' \Delta m_{ee}^2 \left(\frac{K_1^{\alpha\beta}}{\Delta\lambda_{31}} - \frac{K_2^{\alpha\beta}}{\Delta\lambda_{32}} \right) .$$

$$K_1^{\alpha\beta} = \begin{cases} 0 & \alpha = \beta \\ \mp s_{23} c_{23} c_\phi s_\psi^2 (c_\phi^2 c_\psi^2 - s_\phi^2) \sin\delta & \alpha \neq \beta \end{cases} ,$$

where the minus sign is for $\nu_\mu \to \nu_e$.

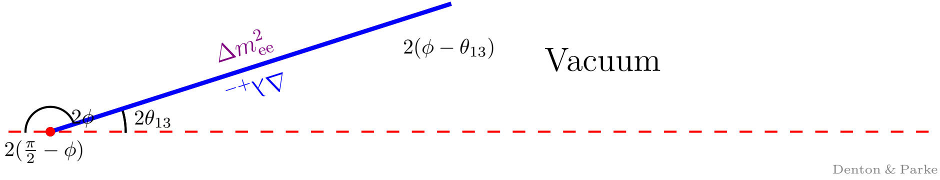
First order coefficients

$\nu_\alpha \to \nu_\beta$	$F_1^{\alpha\beta}$
$\nu_e \to \nu_e$	$-2c_\phi^3 s_\phi s_\psi^3 c_\psi$
$\nu_\mu \to \nu_e$	$c_\phi s_\psi^2[s_\phi s_\psi c_\psi(c_{23}^2 + c_{2\phi}s_{23}^2)$ $-s_{23}c_{23}(s_\phi^2 s_\psi^2 + c_{2\phi}c_\psi^2)\cos\delta]$
$\nu_\mu \to \nu_\mu$	$2c_\phi s_\psi(s_{23}^2 s_\phi c_\psi + s_{23}c_{23}s_\psi \cos\delta)\times$ $(c_{23}^2 c_\psi^2 - 2s_{23}c_{23}s_\phi s_\psi c_\psi \cos\delta + s_{23}^2 s_\phi^2 s_\psi^2)$

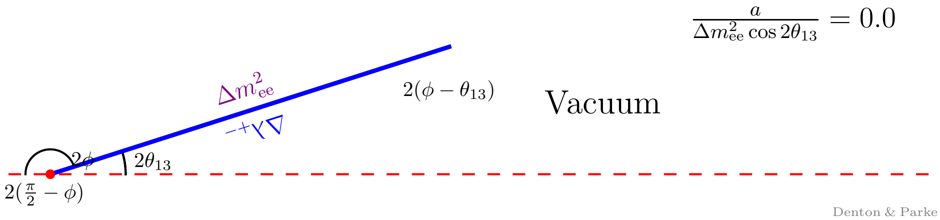
$\nu_\alpha \to \nu_\beta$	$G_1^{\alpha\beta}$
$\nu_e \to \nu_e$	$2s_\phi c_\phi s_\psi c_\psi c_{2\phi}$
$\nu_\mu \to \nu_e$	$-2s_\phi c_\phi s_\psi(s_{23}^2 c_{2\phi}c_\psi - s_{23}c_{23}s_\phi s_\psi \cos\delta)$
$\nu_\mu \to \nu_\mu$	$-2c_\phi s_\psi(s_{23}^2 s_\phi c_\psi + s_{23}c_{23}s_\psi \cos\delta)$ $\times(1 - 2c_\phi^2 s_{23}^2)$

31 Triangle

$$\frac{a}{\Delta m^2_{ee}\cos 2\theta_{13}} = 0.0$$



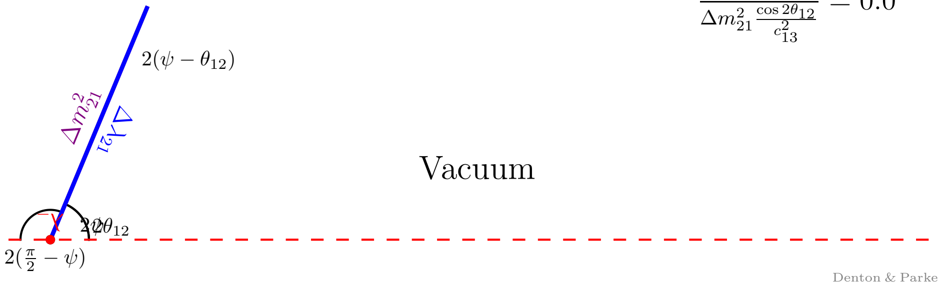
31 Triangle



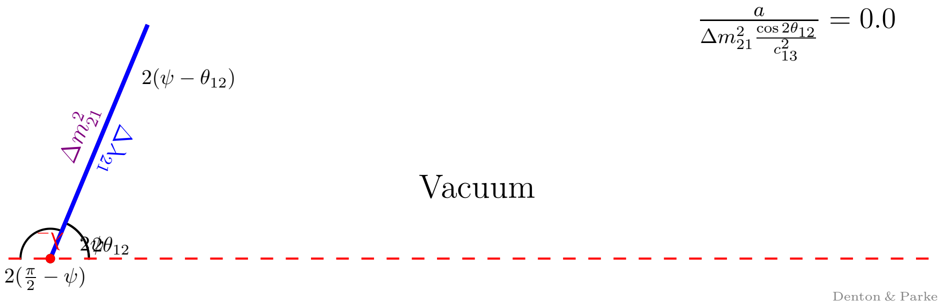
$\frac{a}{\Delta m^2_{ee}\cos 2\theta_{13}} = 0.0$
Δm^2_{ee}
$2(\phi - \theta_{13})$
Vacuum
2ϕ
$2\theta_{13}$
$2(\frac{\pi}{2} - \phi)$
Denton & Parke

21 Triangle

$$\frac{a}{\Delta m^2_{21}\frac{\cos 2\theta_{12}}{c^2_{13}}} = 0.0$$



21 Triangle



$\frac{a}{\Delta m^2_{21}\frac{\cos 2\theta_{12}}{c^2_{13}}} = 0.0$

$2(\psi - \theta_{12})$

Δm^2_{21}

$\Delta\lambda_{21}$

Vacuum

$-\chi$

2ψ

$2\theta_{12}$

$2(\frac{\pi}{2} - \psi)$

Denton & Parke

CPV term

It is known that the exact CPV term in matter is

$$P \supset \pm 8 \sin\delta c_{12} s_{12} c_{13}^2 s_{13} c_{23} s_{23} \prod_{i>j} \frac{\Delta m_{ij}^2}{\Delta\lambda_{ij}} \sin\Delta_{32}^m \sin\Delta_{31}^m \sin\Delta_{21}^m \,.$$

V. Naumov, Int. J. Mod. Phys. 1992

P. Harrison, W. Scott, hep-ph/9912435

Our expression reproduces this order by order in ϵ'.

Precision

At the first oscillation minimum and maximum:

DUNE: NO, $\delta = 3\pi/2$		First min	First max
$P(\nu_\mu \to \nu_e)$		0.0047	0.081
E (GeV)		1.2	2.2
$\frac{\lvert\Delta P\rvert}{P}$	Zeroth	5×10^{-4}	4×10^{-4}
	First	3×10^{-7}	2×10^{-7}
	Second	6×10^{-10}	5×10^{-10}

Conclusions

- Matter effects provide a rich environment to test the neutrino sector.
- Analytic expression for neutrino oscillations with precision and clarity.
- Clear that exactly two matter rotations is the correct number.
- Our perturbative parameter is < 0.015 and is 0 in vacuum.
- Form emphasizes the L/E dependence of oscillations at all orders.
- Zeroth order expression matches the form of the vacuum expression.
- Zeroth order is sufficient for current and planned experiments.
- Code is available on github at github.com/PeterDenton/Nu-Pert.

Backups

Note on the PMNS matrix

The PMNS matrix typically has the CP phase associated with θ_{13}, while our unitary matrix has δ_{CP} associated with θ_{23}. The standard form is related to our form by multiplying the third row by $e^{i\delta}$ and the third column by $e^{-i\delta}$.

Our formulation is useful because after the U_{23} rotation the resulting Hamiltonian of the "tilde" basis is real.

B. Pontecorvo, Sov. Phys. JETP 7 1958

Z. Maki, M. Nakagawa, S. Sakata, Prog. Theor. Phys. 28 1962

A constant rotation in the $2-3$ plane

Change basis by rotating $H^m \to \tilde{H}^m$ by $U_{23}(\theta_{23}, \delta)$

- U_{23} commutes with the matter potential,
- $\tilde{H}^m$ is now entirely real.

$$\begin{aligned}
\lambda_a &= a + (s_{13}^2 + \epsilon s_{12}^2)\Delta m_{ee}^2 \\
\lambda_b &= \epsilon c_{12}^2 \Delta m_{ee}^2 \\
\lambda_c &= (c_{13}^2 + \epsilon s_{12}^2)\Delta m_{ee}^2
\end{aligned}$$

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$$\lambda_b = \epsilon c_{12}^2 \Delta m_{ee}^2$$
$$\lambda_c = (c_{13}^2 + \epsilon s_{12}^2)\Delta m_{ee}^2$$

$$\Delta m_{ee}^2 = \Delta m_{31}^2 - s_{12}^2 \Delta m_{21}^2$$
$$\epsilon = \frac{\Delta m_{21}^2}{\Delta m_{ee}^2} \approx 0.03$$

H. Nunokawa, S. Parke, R. Zukanovich Funchal, hep-ph/0503283

A rotation in the $1-3$ plane

Change basis again by rotating $\tilde{H}^m \to \hat{H}^m$ by $U_{13}(\phi)$

- $\phi(a=0)=\theta_{13}$.

$$\lambda_\mp = \frac{1}{2}\left[(\lambda_a+\lambda_c)\mp\mathrm{sign}(\Delta m^2_{ee})\sqrt{(\lambda_c-\lambda_a)^2+4(s_{13}c_{13}\Delta m^2_{ee})^2}\right]$$

$$\lambda_0=\lambda_b$$

A rotation in the $1-2$ plane

Change basis again by rotating $\hat{H}^m \to \check{H}^m$ by $U_{12}(\psi)$

- $\psi(a=0)=\theta_{12}$,
- We have now created our own $U^m(\theta_{23},\delta,\phi,\psi)$.

$$\lambda_{1,2}=\frac{1}{2}\left[(\lambda_0+\lambda_-)\mp\sqrt{(\lambda_0-\lambda_-)^2+4(\epsilon c_{(\phi-\theta_{13})}c_{12}s_{12}\Delta m^2_{ee})^2}\right]$$
$$\lambda_3=\lambda_+$$

Neutrino oscillation probabilities

$$\mathcal{A}(\nu_\alpha \to \nu_\beta) = \sum_{i=1}^{3} U_{\alpha i}^* U_{\beta i} e^{-i\frac{m_i^2 L}{2E}}$$

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$$\mathcal{A}(\nu_\alpha \to \nu_\beta) = \sum_{i=1}^{3} U_{\alpha i}^* U_{\beta i} e^{-i\frac{m_i^2 L}{2E}}$$

$$\begin{aligned} P(\nu_\alpha \to \nu_\beta) = & \left|\mathcal{A}(\nu_\alpha \to \nu_\beta)\right|^2 \\ = & \delta_{\alpha\beta} - \sum_{i<j} \Re\left[U_{\alpha i} U_{\beta i}^* U_{\alpha j}^* U_{\beta j}\right] \sin^2 \Delta_{ij} \\ & + 8\Im\left[U_{\alpha 1} U_{\beta 2}^* U_{\alpha 2}^* U_{\beta 1}\right] \sin\Delta_{32} \sin\Delta_{31} \sin\Delta_{21} \end{aligned}$$

C. Jarlskog, PRL 1985

Nonvanishing Wronskian $\Rightarrow$ fewest number of L/E functions.

$$\Delta_{ij} = \frac{\Delta m_{ij}^2 L}{4E}$$

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C. Jarlskog, PRL 1985

Nonvanishing Wronskian $\Rightarrow$ fewest number of L/E functions.

Clear that the CPV term is $\mathcal{O}[(L/E)^3]$ not $\mathcal{O}[(L/E)^1]$.

$$\Delta_{ij} = \frac{\Delta m_{ij}^2 L}{4E}$$