Abstract

We further develop and extend a recent perturbative framework for neutrino oscillations in uniform matter density so that the resulting oscillation probabilities are accurate for the complete matter potential versus baseline divided by neutrino energy plane. This extension also gives the *exact* oscillation probabilities in vacuum for all values of baseline divided by neutrino energy. The expansion parameter used is related to the ratio of the solar to the atmospheric Δm^2 scales but with a unique choice of the atmospheric Δm^2 such that certain first-order effects are taken into account in the zeroth-order Hamiltonian. Using a mixing matrix formulation, this framework has the exceptional feature that the neutrino oscillation probability in matter has the same structure as in vacuum, to all orders in the expansion parameter. It also contains all orders in the matter potential and $\sin\theta_{13}$. It facilitates immediate physical interpretation of the analytic results, and makes the expressions for the neutrino oscillation probabilities extremely compact and very accurate even at zeroth order in our perturbative expansion. The first and second order results are also given which improve the precision by approximately two or more orders of magnitude per perturbative order.

Analytic and compact perturbative expressions for neutrino oscillations in matter

Peter B. Denton

Pheno 2016

May 9, 2016

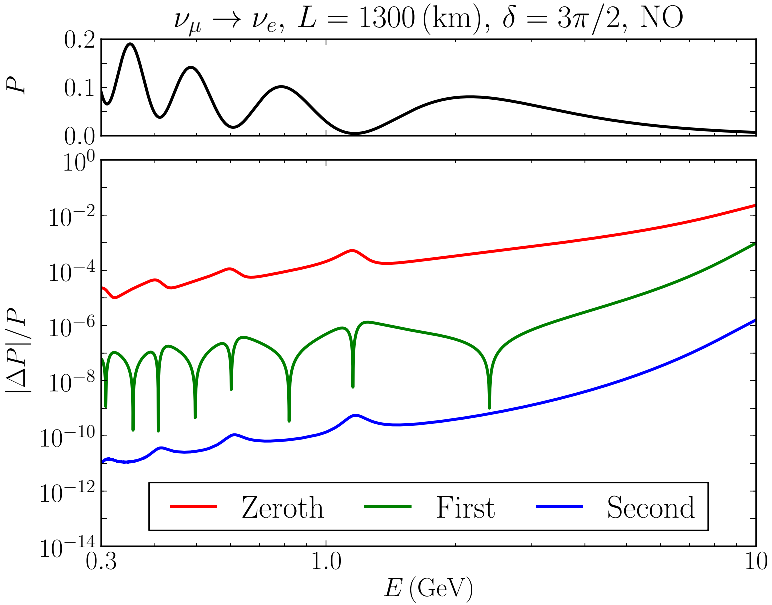
Work done with S. Parke and H. Minakata.
1604.08167
github.com/PeterDenton/Nu-Pert

A perturbative description of neutrino oscillations in matter

Several goals:

1. Use $\epsilon \simeq \Delta m^2_{21}/\Delta m^2_{31} \approx 0.03$ as an expansion parameter.
2. Minimize the number of additional 'matter' terms (2).
3. Extremely accurate for all channels.
4. Exact in the vacuum limit.
5. Reproduce the known CPV term in matter.

Precision order-by-order



Neutrino oscillations in matter

The electron flavor interacts differently in matter than the others through a G_F type process.

$$H^m = \frac{1}{2E}\left[U\begin{pmatrix} m_1^2 & & \\ & m_2^2 & \\ & & m_3^2 \end{pmatrix}U^\dagger + \begin{pmatrix} a & & \\ & & \\ & & \end{pmatrix}\right]$$

We could write

$$H^m = \frac{1}{2E}U^m\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}U^{m\dagger}$$

where the λ_i are the masses squared in matter
and $\theta_{12}^m, \theta_{13}^m, \theta_{23}^m$, and δ^m and the angles and phase in matter.

$$U \equiv U_{23}(\theta_{23},\delta)U_{13}(\theta_{13})U_{12}(\theta_{12})$$

Exact neutrino oscillations in matter: eigenvalues

This has been done.

H. Zaglauer, K. Schwarzer, Z. Phys. C Particles and Fields, 40 1988

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$$\lambda_1 = \frac{A}{3} - \frac{1}{3}\sqrt{A^2 - 3B}S - \frac{\sqrt{3}}{3}\sqrt{A^2 - 3B}\sqrt{1 - S^2}$$
$$\lambda_2 = \frac{A}{3} - \frac{1}{3}\sqrt{A^2 - 3B}S + \frac{\sqrt{3}}{3}\sqrt{A^2 - 3B}\sqrt{1 - S^2}$$
$$\lambda_3 = \frac{A}{3} + \frac{2}{3}\sqrt{A^2 - 3B}S$$

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$$A = \Delta m^2_{21} + \Delta m^2_{31} + a$$

$$B = \Delta m^2_{21}\Delta m^2_{31} + a\left[c^2_{13}\Delta m^2_{31} + (c^2_{12}c^2_{13} + s^2_{13})\Delta m^2_{21}\right]$$

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$$S = \cos\left\{\frac{1}{3}\cos^{-1}\left[\frac{2A^3 - 9AB + 27C}{2\left(A^2 - 3B\right)^{3/2}}\right]\right\}$$

$$C = a\Delta m^2_{21}\Delta m^2_{31}c^2_{12}c^2_{13}$$

Alternative solutions

- Numerical methods.
  - Good for experiments.
  - Important understanding can be missed (magic baseline)

P. Huber, W. Winter, hep-ph/0301257

A. Smirnov, hep-ph/0610198

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A. Smirnov, hep-ph/0610198

- Perturbative expansion
  - Small matter potential: $a/\Delta m^2$.
  - s_{13}, s_{13}^2.

A. Cervera, et. al., hep-ph/0002108

H. Minakata, 0910.5545

K. Asano, H. Minakata, 1103.4387

  - $\Delta m^2_{21}/\Delta m^2_{31} \sim 0.03$.

J. Arafune, J. Sato, hep-ph/9607437

A. Cervera, et. al., hep-ph/0002108

M. Freund, hep-ph/0103300

E. Akhmedov, et. al., hep-ph/0402175

H. Minakata, S. Parke, 1505.01826

PBD, H. Minakata, S. Parke, 1604.08167

Our methodology

- Start with $\epsilon = \frac{\Delta m^2_{21}}{\Delta m^2_{ee}} = 0.03$.

$$\Delta m^2_{ee} = \Delta m^2_{31} - s^2_{12}\Delta m^2_{21}$$

H. Nunokawa, S. Parke, R. Zukanovich Funchal, hep-ph/0503283

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- Write the probabilities with simple L/E dependence:

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - \sum_{i<j} \Re\left[U_{\alpha i} U^*_{\beta j} U^*_{\alpha j} U_{\beta i}\right] \sin^2 \Delta_{ij}$$
$$+ 8\Im\left[U_{\alpha 1} U^*_{\beta 2} U^*_{\alpha 2} U_{\beta 1}\right] \sin \Delta_{32} \sin \Delta_{31} \sin \Delta_{21}$$

C. Jarlskog, PRL 1985

Nonvanishing Wronskian $\Rightarrow$ fewest number of L/E functions.

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$$\Delta_{ij} = \frac{\Delta m^2_{ij} L}{4E}$$

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Nonvanishing Wronskian $\Rightarrow$ fewest number of L/E functions.

Clear that the CPV term is $\mathcal{O}[(L/E)^3]$ not $\mathcal{O}[(L/E)^1]$.

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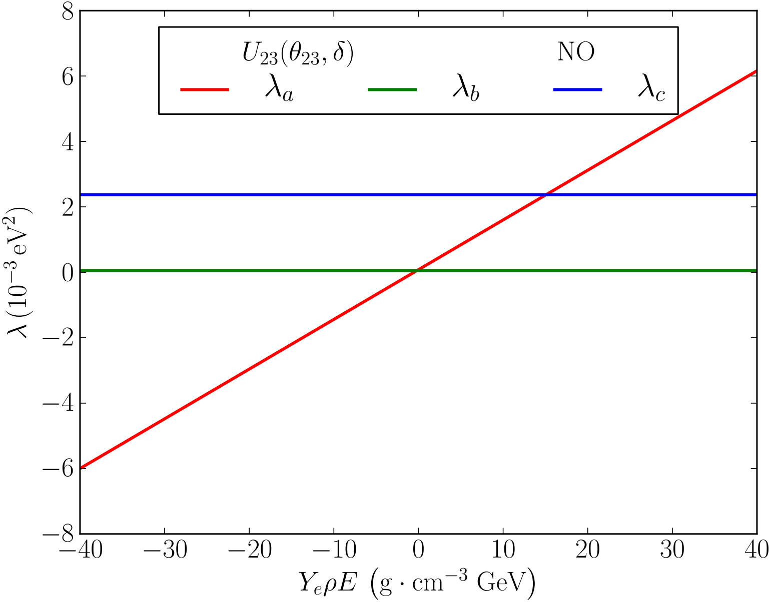
Eigenvalues in matter: two rotations are needed



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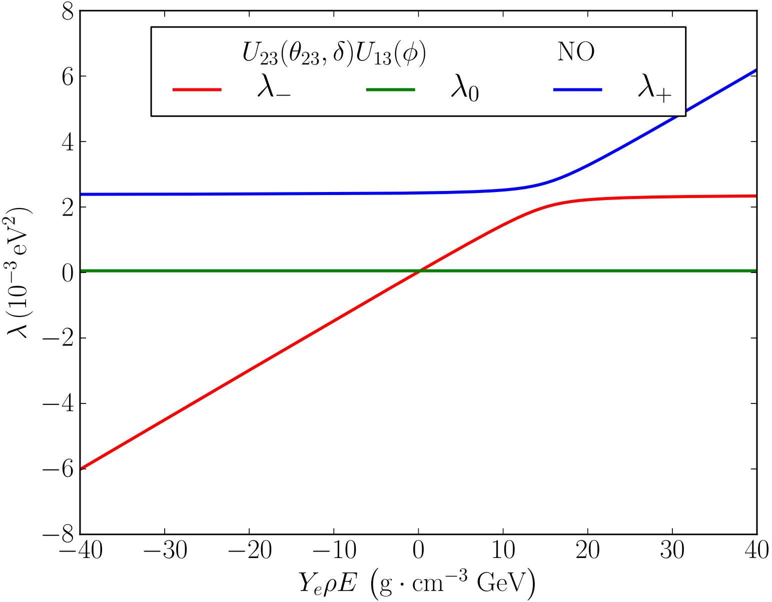

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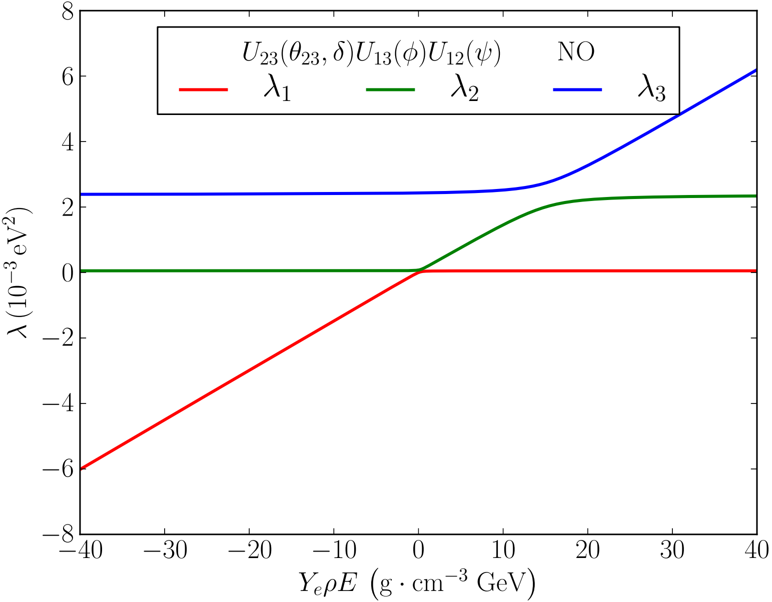

Three rotations

1. Perform a constant $U_{23}(\theta_{23}, \delta)$ rotation.
   - U_{23} commutes with the matter potential.
   - Resultant Hamiltonian is real.
   - 'Expansion parameter' is $c_{13}s_{13} = 0.15$ at this point.

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2. Diagonalize the diagonal and $\mathcal{O}(\epsilon^0)$ off-diagonal terms with $U_{13}(\phi)$.
   - $\phi_{\rm vac} = \theta_{13}$.
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H. Minakata, S. Parke, 1505.01826

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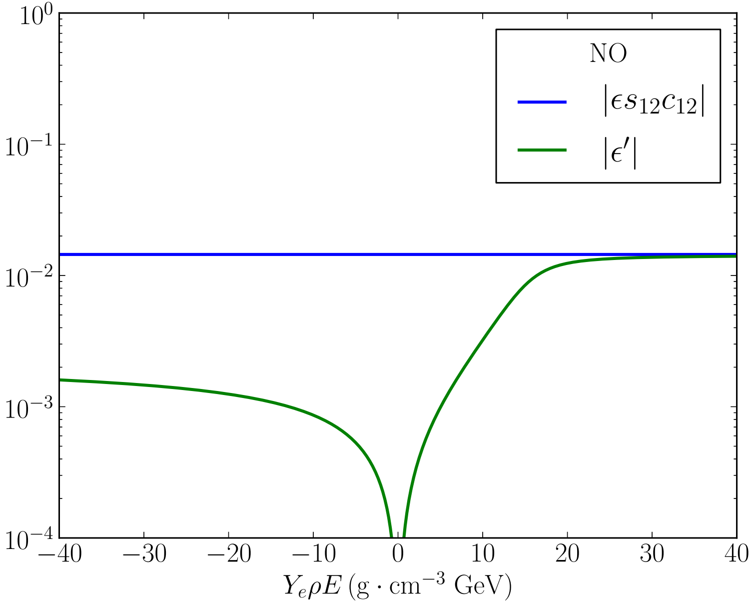
3. Diagonalize the terms non-zero in matter with $U_{12}(\psi)$.
   - $\psi_{\rm vac} = \theta_{12}$.
   - Expansion parameter is now $\epsilon' = c_{12}s_{12}s_{(\phi-\theta_{13})}\frac{\Delta m^2_{21}}{\Delta m^2_{ee}} < 0.015$.

Expansion parameter



Simplicity at zeroth order

Simple L/E dependence:

$$P = \delta^{\alpha\beta} + 4C_{21}^{\alpha\beta}\sin^2\Delta_{21} + 4C_{31}^{\alpha\beta}\sin^2\Delta_{31} + 4C_{32}^{\alpha\beta}\sin^2\Delta_{32}$$
$$+8D^{\alpha\beta}\sin\Delta_{21}\sin\Delta_{31}\sin\Delta_{32}$$

$$\Delta_{ij} = \frac{\Delta m_{ij}^2 L}{4E}$$

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Zeroth order is equivalent to the vacuum probability with

$$\theta_{13} \to \phi$$

$$\theta_{12} \to \psi$$

and

$$\Delta m_{ij}^2 \to \Delta\lambda_{ij}$$

$$\Delta_{ij} = \frac{\Delta m_{ij}^2 L}{4E}$$

CPV term

It is known that the exact CPV term in matter is

$$\pm 8 \sin\delta c_{12} s_{12} c_{13}^2 s_{13} c_{23} s_{23} \prod_{i>j} \frac{\Delta m_{ij}^2}{\Delta\lambda_{ij}^{(\mathrm{ex})}} \sin\Delta_{32} \sin\Delta_{31} \sin\Delta_{21}$$

V. Naumov, Int. J. Mod. Phys. 1992

P. Harrison, W. Scott, `hep-ph/9912435`

Our expression reproduces this order by order in ϵ'.

Precision

At the first oscillation minimum and maximum:

DUNE: NO, $\delta = 3\pi/2$		First min	First max
$P(\nu_\mu \to \nu_e)$		0.0047	0.081
E (GeV)		1.2	2.2
$\frac{\lvert\Delta P\rvert}{P}$	Zeroth	5×10^{-4}	4×10^{-4}
	First	3×10^{-7}	2×10^{-7}
	Second	6×10^{-10}	5×10^{-10}

Conclusions

- Analytic expression for neutrino oscillations with precision and clarity.
- Clear that exactly two matter rotations is the correct number.
- Our perturbative parameter is < 0.015 and is 0 in vacuum.
- Form emphasizes the L/E dependence of oscillations at all orders.
- Zeroth order expression matches the form of the vacuum expression.
- Zeroth order is sufficient for current and planned experiments.

Backups

Eigenvalues

Tilde basis after $U_{23}(\theta_{23}, \delta)$:

$$\lambda_a = a + (s_{13}^2 + \epsilon s_{12}^2)\Delta m_{ee}^2$$
$$\lambda_b = \epsilon c_{12}^2 \Delta m_{ee}^2$$
$$\lambda_c = (c_{13}^2 + \epsilon s_{12}^2)\Delta m_{ee}^2$$

Hat basis after $U_{13}(\phi)$:

$$\lambda_\mp = \frac{1}{2}\left[(\lambda_a + \lambda_c) \mp \text{sign}(\Delta m_{ee}^2)\sqrt{(\lambda_c - \lambda_a)^2 + 4(s_{13}c_{13}\Delta m_{ee}^2)^2}\right]$$
$$\lambda_0 = \lambda_b$$

Check basis after $U_{12}(\psi)$:

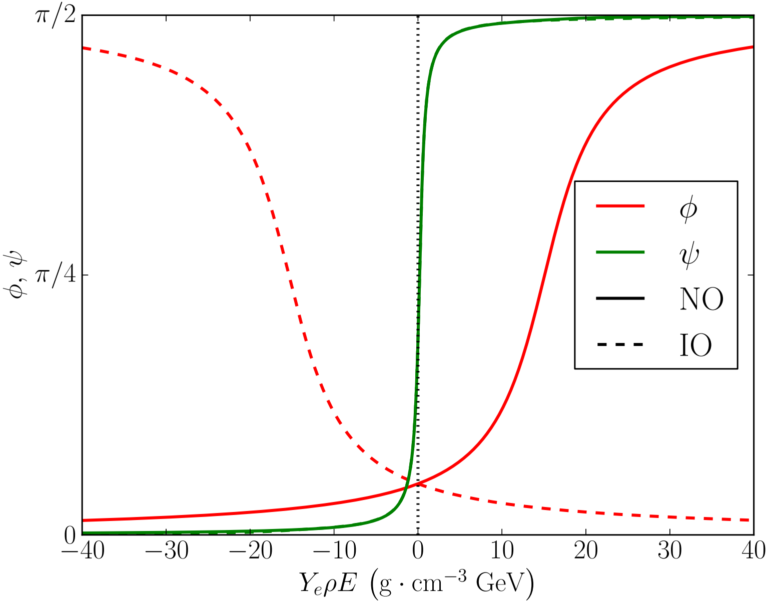
$$\lambda_{1,2} = \frac{1}{2}\left[(\lambda_0 + \lambda_-) \mp \sqrt{(\lambda_0 - \lambda_-)^2 + 4(\epsilon c_{(\phi-\theta_{13})} c_{12} s_{12} \Delta m_{ee}^2)^2}\right]$$
$$\lambda_3 = \lambda_+$$

Phases

$$c_{2\phi} = \frac{\lambda_c - \lambda_a}{\lambda_+ - \lambda_-} \qquad s_{2\phi} = s_{2\theta_{13}} \frac{\Delta m^2_{ee}}{\lambda_+ - \lambda_-}$$

$$c_{2\psi} = \frac{\lambda_0 - \lambda_-}{\lambda_2 - \lambda_1} \qquad s_{2\psi} = s_{2\theta_{12}} \epsilon c_{(\phi - \theta_{13})} \frac{\Delta m^2_{ee}}{\lambda_2 - \lambda_1}$$

The two phases in matter



$\lambda_{1,2} - \psi$ interchange

From the shape of $U_{12}(\psi)$, it is clear that the probabilities are invariant under a simultaneous interchange of

$$\lambda_1 \leftrightarrow \lambda_2 \qquad \text{and} \qquad \psi \to \psi \pm \frac{\pi}{2}$$

Since only even powers of ψ trig functions $(c_\psi^2, s_\psi^2, c_\psi s_\psi, c_{2\psi}, s_{2\psi})$ appear in the probabilities, the sign degeneracy is irrelevant. More usefully, we can write that the probabilities are invariant under the simultaneous interchange of

$$\lambda_1 \leftrightarrow \lambda_2 \qquad c_\psi^2 \leftrightarrow s_\psi^2 \qquad \text{and} \qquad c_\psi s_\psi \to -c_\psi s_\psi$$

This interchange constrains C_{21},
and C_{32} is then easily calculated from C_{31}.

Zeroth order coefficients

$\nu_\alpha \to \nu_\beta$	$(C_{21}^{\alpha\beta})^{(0)}$
$\nu_e \to \nu_e$	$-c_\phi^4 s_\psi^2 c_\psi^2$
$\nu_\mu \to \nu_e$	$c_\phi^2 s_\psi^2 c_\psi^2 (c_{23}^2 - s_\phi^2 s_{23}^2) + c_{2\psi} J_r^m \cos\delta$
$\nu_\mu \to \nu_\mu$	$-(c_{23}^2 c_\psi^2 + s_{23}^2 s_\phi^2 s_\psi^2)(c_{23}^2 s_\psi^2 + s_{23}^2 s_\phi^2 c_\psi^2)$ $-2(c_{23}^2 - s_\phi^2 s_{23}^2) c_{2\psi} J_{rr}^m \cos\delta + (2J_{rr}^m \cos\delta)^2$

$\nu_\alpha \to \nu_\beta$	$(C_{31}^{\alpha\beta})^{(0)}$	$(D^{\alpha\beta})^{(0)}$
$\nu_e \to \nu_e$	$-c_\phi^2 s_\phi^2 c_\psi^2$	0
$\nu_\mu \to \nu_e$	$s_\phi^2 c_\phi^2 c_\psi^2 s_{23}^2 + J_r^m \cos\delta$	$-J_r^m \sin\delta$
$\nu_\mu \to \nu_\mu$	$-c_\phi^2 s_{23}^2 (c_{23}^2 s_\psi^2 + s_{23}^2 s_\phi^2 c_\psi^2)$ $-2 s_{23}^2 J_r^m \cos\delta$	0

$$J_r^m \equiv s_\psi c_\psi s_\phi c_\phi^2 s_{23} c_{23}$$
$$J_{rr}^m \equiv J_r^m / c_\phi^2$$

General form of the first order coefficients

$$(C_{21}^{\alpha\beta})^{(1)} = \epsilon' \Delta m_{ee}^2 \left(\frac{F_1^{\alpha\beta}}{\Delta\lambda_{31}} + \frac{F_2^{\alpha\beta}}{\Delta\lambda_{32}} \right)$$

$$(C_{31}^{\alpha\beta})^{(1)} = \epsilon' \Delta m_{ee}^2 \left(\frac{F_1^{\alpha\beta} + G_1^{\alpha\beta}}{\Delta\lambda_{31}} - \frac{F_2^{\alpha\beta}}{\Delta\lambda_{32}} \right)$$

$$(C_{32}^{\alpha\beta})^{(1)} = \epsilon' \Delta m_{ee}^2 \left(-\frac{F_1^{\alpha\beta}}{\Delta\lambda_{31}} + \frac{F_2^{\alpha\beta} + G_2^{\alpha\beta}}{\Delta\lambda_{32}} \right)$$

$$(D^{\alpha\beta})^{(1)} = \epsilon' \Delta m_{ee}^2 \left(\frac{K_1^{\alpha\beta}}{\Delta\lambda_{31}} - \frac{K_2^{\alpha\beta}}{\Delta\lambda_{32}} \right)$$

$$K_1^{\alpha\beta} = \begin{cases} 0 & \alpha = \beta \\ \mp s_{23} c_{23} c_\phi s_\psi^2 (c_\phi^2 c_\psi^2 - s_\phi^2) \sin\delta & \alpha \neq \beta \end{cases}$$

where the minus sign is for $\nu_\mu \to \nu_e$.

First order coefficients

$\nu_\alpha \to \nu_\beta$	$F_1^{\alpha\beta}$
$\nu_e \to \nu_e$	$-2c_\phi^3 s_\phi s_\psi^3 c_\psi$
$\nu_\mu \to \nu_e$	$c_\phi s_\psi^2[s_\phi s_\psi c_\psi(c_{23}^2 + c_{2\phi}s_{23}^2)$ $-s_{23}c_{23}(s_\phi^2 s_\psi^2 + c_{2\phi}c_\psi^2)\cos\delta]$
$\nu_\mu \to \nu_\mu$	$2c_\phi s_\psi(s_{23}^2 s_\phi c_\psi + s_{23}c_{23}s_\psi\cos\delta)\times$ $(c_{23}^2 c_\psi^2 - 2s_{23}c_{23}s_\phi s_\psi c_\psi\cos\delta + s_{23}^2 s_\phi^2 s_\psi^2)$

$\nu_\alpha \to \nu_\beta$	$G_1^{\alpha\beta}$
$\nu_e \to \nu_e$	$2s_\phi c_\phi s_\psi c_\psi c_{2\phi}$
$\nu_\mu \to \nu_e$	$-2s_\phi c_\phi s_\psi(s_{23}^2 c_{2\phi}c_\psi - s_{23}c_{23}s_\phi s_\psi\cos\delta)$
$\nu_\mu \to \nu_\mu$	$-2c_\phi s_\psi(s_{23}^2 s_\phi c_\psi + s_{23}c_{23}s_\psi\cos\delta)$ $\times(1-2c_\phi^2 s_{23}^2)$

Note on the PMNS matrix

The PMNS matrix typically has the CP phase associated with θ_{13}, while our unitary matrix has δ_{CP} associated with θ_{23}. The standard form is related to our form by multiplying the third row by $e^{i\delta}$ and the third column by $e^{-i\delta}$.

Our formulation is useful because after the U_{23} rotation the resulting Hamiltonian of the "tilde" basis is real.

B. Pontecorvo, Sov. Phys. JETP 7 1958

Z. Maki, M. Nakagawa, S. Sakata, Prog. Theor. Phys. 28 1962

Neutrino oscillations in vacuum

Neutrino oscillations in matter are described by this Hamiltonian written in the flavor basis:

$$H = \frac{1}{2E} U \begin{pmatrix} m_1^2 & & \\ & m_2^2 & \\ & & m_3^2 \end{pmatrix} U^\dagger$$

where the unitary mixing matrix is parameterized by

$$U = \begin{pmatrix} 1 & & \\ & c_{23} & s_{23}e^{i\delta} \\ & -s_{23}e^{-i\delta} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & & s_{13} \\ & 1 & \\ -s_{13} & & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & \\ -s_{12} & c_{12} & \\ & & 1 \end{pmatrix}$$

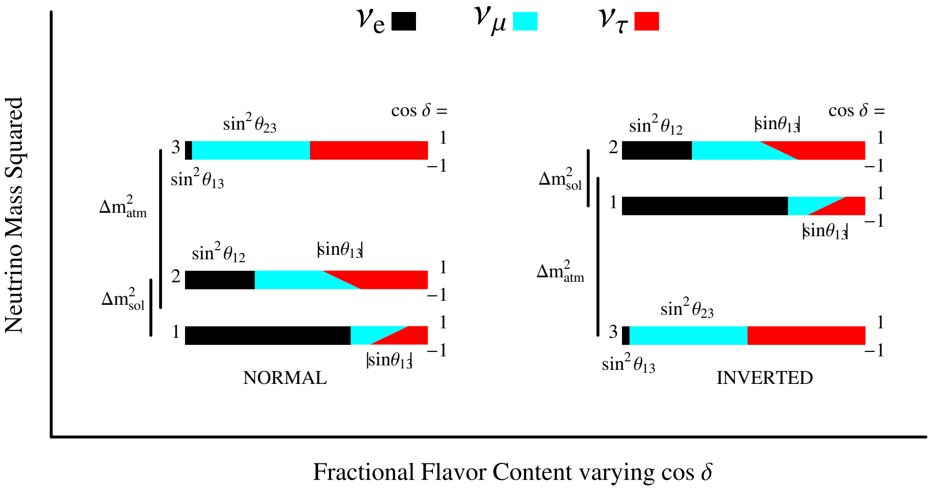
$$= \begin{pmatrix} c_{12}c_{13} & c_{13}s_{12} & s_{13} \\ -c_{23}s_{12} - e^{i\delta}c_{12}s_{13}s_{23} & c_{12}c_{23} - e^{i\delta}s_{12}s_{13}s_{23} & e^{i\delta}c_{13}s_{23} \\ e^{-i\delta}s_{12}s_{23} - c_{12}c_{23}s_{13} & -c_{23}s_{12}s_{13} - e^{-i\delta}c_{12}s_{23} & c_{13}c_{23} \end{pmatrix}$$

Neutrino oscillations in vacuum: example

For example, it is easy to calculate the *exact* $\nu_e \to \nu_e$ disappearance channel in vacuum.

$$\begin{aligned} P(\nu_e \to \nu_e) = 1 \\ & - 4c_{12}^2 s_{12}^2 c_{13}^4 \sin^2 \Delta_{21} \\ & - 4c_{12}^2 c_{13}^2 s_{13}^2 \sin^2 \Delta_{31} \\ & - 4s_{12}^2 c_{13}^2 s_{13}^2 \sin^2 \Delta_{32} \end{aligned}$$

Known neutrino properties



O. Mena, S. Parke, `hep-ph/0312131`

A constant rotation in the $2-3$ plane

Change basis by rotating $H^m \to \tilde{H}^m$ by $U_{23}(\theta_{23}, \delta)$

- U_{23} commutes with the matter potential,
- $\tilde{H}^m$ is now entirely real.

$$\lambda_a = a + (s_{13}^2 + \epsilon s_{12}^2)\Delta m_{ee}^2$$
$$\lambda_b = \epsilon c_{12}^2 \Delta m_{ee}^2$$
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$$\Delta m_{ee}^2 = \Delta m_{31}^2 - s_{12}^2 \Delta m_{21}^2$$
$$\epsilon = \frac{\Delta m_{21}^2}{\Delta m_{ee}^2} \approx 0.03$$

H. Nunokawa, S. Parke, R. Zukanovich Funchal, `hep-ph/0503283`

A rotation in the $1-3$ plane

Change basis again by rotating $\tilde{H}^m \to \hat{H}^m$ by $U_{13}(\phi)$

- $\phi(a=0)=\theta_{13}$.

$$\lambda_\mp = \frac{1}{2}\left[(\lambda_a+\lambda_c)\mp\text{sign}(\Delta m^2_{ee})\sqrt{(\lambda_c-\lambda_a)^2+4(s_{13}c_{13}\Delta m^2_{ee})^2}\right]$$

$$\lambda_0=\lambda_b$$

A rotation in the $1-2$ plane

Change basis again by rotating $\hat{H}^m \to \check{H}^m$ by $U_{12}(\psi)$

- $\psi(a=0)=\theta_{12}$,
- We have now created our own $U^m(\theta_{23},\delta,\phi,\psi)$.

$$\lambda_{1,2} = \frac{1}{2}\left[(\lambda_0+\lambda_-) \mp \sqrt{(\lambda_0-\lambda_-)^2 + 4(\epsilon c_{(\phi-\theta_{13})} c_{12} s_{12} \Delta m^2_{ee})^2}\right]$$
$$\lambda_3 = \lambda_+$$

Exact neutrino oscillations in matter: mixing angles

$$s_{12}^{m2}=\frac{-\left(\lambda_2^2-\alpha\lambda_2+\beta\right)\Delta\lambda_{31}}{\left(\lambda_1^2-\alpha\lambda_1+\beta\right)\Delta\lambda_{32}-\left(\lambda_2^2-\alpha\lambda_2+\beta\right)\Delta\lambda_{31}}$$

$$s_{13}^{m2}=\frac{\lambda_3^2-\alpha\lambda_3+\beta}{\Delta\lambda_{31}\Delta\lambda_{32}}$$

$$s_{23}^{m2}=\frac{s_{23}^2E^2+c_{23}^2F^2+2c_{23}s_{23}c_\delta EF}{E^2+F^2}$$

$$e^{-i\delta^m}=\frac{c_{23}^2s_{23}^2\left(e^{-i\delta}E^2-e^{i\delta}F^2\right)+\left(c_{23}^2-s_{23}^2\right)EF}{\sqrt{\left(s_{23}^2E^2+c_{23}^2F^2+2EFc_{23}s_{23}c_\delta\right)\left(c_{23}^2E^2+s_{23}^2F^2-2EFc_{23}s_{23}c_\delta\right)}}$$

Exact neutrino oscillations in matter: mixing angles

$$s_{12}^{m2} = \frac{-\left(\lambda_2^2 - \alpha\lambda_2 + \beta\right)\Delta\lambda_{31}}{\left(\lambda_1^2 - \alpha\lambda_1 + \beta\right)\Delta\lambda_{32} - \left(\lambda_2^2 - \alpha\lambda_2 + \beta\right)\Delta\lambda_{31}}$$

$$s_{13}^{m2} = \frac{\lambda_3^2 - \alpha\lambda_3 + \beta}{\Delta\lambda_{31}\Delta\lambda_{32}}$$

$$s_{23}^{m2} = \frac{s_{23}^2 E^2 + c_{23}^2 F^2 + 2c_{23}s_{23}c_\delta EF}{E^2 + F^2}$$

$$e^{-i\delta^m} = \frac{c_{23}^2 s_{23}^2 \left(e^{-i\delta}E^2 - e^{i\delta}F^2\right) + \left(c_{23}^2 - s_{23}^2\right)EF}{\sqrt{\left(s_{23}^2E^2 + c_{23}^2F^2 + 2EFc_{23}s_{23}c_\delta\right)\left(c_{23}^2E^2 + s_{23}^2F^2 - 2EFc_{23}s_{23}c_\delta\right)}}$$

$$\alpha = c_{13}^2\Delta m_{31}^2 + \left(c_{12}^2c_{13}^2 + s_{13}^2\right)\Delta m_{21}^2$$

$$\beta = c_{12}^2c_{13}^2\Delta m_{21}^2\Delta m_{31}^2$$

$$E = c_{13}s_{13}\left[\left(\lambda_3 - \Delta m_{21}^2\right)\Delta m_{31}^2 - s_{12}^2\left(\lambda_3 - \Delta m_{31}^2\right)\Delta m_{21}^2\right]$$

$$F = c_{12}s_{12}c_{13}\left(\lambda_3 - \Delta m_{31}^2\right)\Delta m_{21}^2$$

Neutrino oscillation probabilities

$$\mathcal{A}(\nu_\alpha \to \nu_\beta) = \sum_{i=1}^{3} U_{\alpha i}^* U_{\beta i} e^{-i\frac{m_i^2 L}{2E}}$$

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C. Jarlskog, PRL 1985

Nonvanishing Wronskian $\Rightarrow$ fewest number of L/E functions.

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Neutrino oscillation probabilities

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Clear that the CPV term is $\mathcal{O}[(L/E)^3]$ not $\mathcal{O}[(L/E)^1]$.

$$\Delta_{ij} = \frac{\Delta m^2_{ij} L}{4E}$$

Eigenvalues in matter